(**Q1**) Consider the logistic map

$$x_{n+1} = F(x_n) = \mu x_n(1 - x_n), \qquad \text{where } \mu > 0 \quad \text{and} \quad x \in [0, 1].$$

(a) Find the fixed points of this map. For which values of μ do they exist?

(b) Find a period-2 orbit of the map (i.e., find x_0 and $x_1 \neq x_0$ such that $x_1 = F(x_0)$ and $F(x_1) = x_0$). For which values of μ does it exist?

Solution.

(a) x is a fixed point of F iff $x = F(x)$ iff $x = \mu x(1-x)$. Clearly $x = 0$ is one solution, and the other satisfies $1 = \mu(1-x)$ or $x = \frac{-1+\mu}{\mu}$. Since $0 \le x \le 1$, we require that $\mu \ge 1$ for the second fixed point to exist (when $\mu = 1$ it coincides with $x = 0$).

(b) x is a period-2 periodic point iff x is a fixed point of F^2 iff $x = F(F(x))$. There are two obvious solutions: $x = 0$ and $x = \mu x(1-x)$. If we divide the polynomial $F(F(x)) - x = \mu^2 x(1-x)(1-\mu x(1-x)) - x$ by $\mu x(x - \frac{-1+\mu}{\mu})$ we get $\mu^2 x^2 - \mu(\mu+1)x + \mu + 1 = 0$. The quadratic root formula gives

$$x_\pm = \frac{\mu + 1 \pm \sqrt{(\mu-3)(\mu+1)}}{2\mu}.$$

Clearly, these points are real only if $\mu \ge 3$ (and at $\mu = 3$, they equal $2/3 = \left.\frac{-1+\mu}{\mu}\right|_{\mu=3}$, which is also a period-1 periodic point). For $\mu > 3$, they are distinct from the period-1 points, so it must be the case that $F(x_-) = x_+$, $F(x_+) = x_-$.

(**Q2**) Show that the discrete system

$$x_{n+1} = \frac{1}{4} - \frac{1}{2a} - a^2 x_n^2$$

is equivalent to the logistic map with $\mu = a$. (Hint: the variable transformation relating the two systems is affine; i.e., $y_n = \alpha x_n + \beta$.)

Solution. Let $y = \alpha x + \beta$ or $x = \frac{y-\beta}{\alpha}$ and let $f(x) = c - a^2x^2$ where $c = \frac{1}{4} - \frac{1}{2a}$, $g(y) = ay(1-y^2)$. We want $x_{n+1} = f(x_n)$ to imply that $y_{n+1} = g(y_n)$. This leads to

$$\begin{aligned} y_{n+1} = \alpha x_{n+1} + \beta &= \alpha c + \beta - a^2 \alpha x_n^2, \\ &= \alpha c + \beta - \frac{a^2}{\alpha^2}\,(y_n^2 - 2\beta y_n + \beta^2), \\ &= \alpha c + \beta - \frac{a^2\beta^2}{\alpha} + \frac{2a^2\beta}{\alpha} y_n - \frac{a^2}{\alpha} y_n^2 \end{aligned}$$

and we want $y_{n+1} = a y_n(1 - y_n)$ so

$$\frac{a^2}{\alpha} = a \implies \alpha = a, \quad 2a\beta = a \implies \beta = 1/2, \quad ac + 1/2 - a/4 = 0 \implies c = \frac{1}{4} - \frac{1}{2a}.$$

Thereofore the equations are consistent and $y = ax + 1/2$.

Remark. Let us formulate the problem in terms of maps. Let $h(x) = \alpha x + \beta$ and let h^{-1} be the inverse of h. We are seeking a solution to the equation

$$g(y) = h \circ f \circ h^{-1}(y) \qquad \text{or} \qquad g \circ h = h \circ f.$$

The first equation says that g is *conjugate* to f by the conjugacy h. Maps are conjugate iff the corresponding dynamical systems can be obtained from one another by a change of coordinates – the change of coordinates map being the conjugacy.

(**Q3**) Show that the map

$$\theta_{n+1} \equiv 2\theta_n \mod 1$$

can be transformed into the logistic map with $\mu = 4$ and $0 \leq x_n \leq 1$ by the change of variable $x_n = \sin^2(\pi\theta_n)$. Find x_0 such that $x_8 = x_0$ but $x_1, \ldots, x_7 \neq x_0$.

Solution. 1. We want to show that $\theta_{n+1} = 2\theta_n \bmod 1$ and $x_n = \sin^2(\pi\theta_n)$ implies $x_{n+1} = 4x_n(1-x_n)$. If $\theta_{n+1} = 2\theta_n \bmod 1$ and $x_n = \sin^2(\pi\theta_n)$, then

$$\begin{aligned}\sin^2(\pi\theta_{n+1}) &= \sin^2(2\pi\theta_n)\\ &= 4\sin^2(\pi\theta_n)\cos^2(\pi\theta_n),\\ &= 4\sin^2(\pi\theta)\ (1-\sin^2(\pi\theta_n))\\ &= 4x_n(1-x_n),\\ &= x_{n+1}.\end{aligned}$$

Solution 2. Let $f(x) = 4x(1-x)$, $g(\theta) = 2\theta \bmod 1$ and $h(\theta) = \sin^2(\pi\theta)$. We want to show that

$$h \circ g = f \circ h.$$

Now

$$\begin{aligned}h \circ g(\theta) &= \sin^2(2\pi\theta)\\ &= 4\sin^2(\pi\theta)(1-\sin^2(\pi\theta)),\\ &= 4h(\theta)(1-h(\theta)),\\ &= f \circ h(\theta).\end{aligned}$$

Note that the calulation comes down to basically the same thing, except that in the second case we always stay in the θ coordinate system.

To find a period-8 orbit of the logistic map, observe that if $\theta_n = \theta_0$, then $x_n = x_0$. Therefore, we see that the period-n orbits of the times-2 map are given by $\theta_n = 2^n\theta_0 = \theta_0 \bmod 1$ or $\theta_0 = \frac{k}{2^n-1} \bmod 1$ for some integer k. In the case $n = 0$, we get $\theta_0 = \frac{k}{255} \bmod 1$ or $\theta_0 = 0, 1/255, 2/255, \ldots, 254/255 \bmod 1$. We see that if $\theta_0 = 1/255$, then $\theta_0, \cdots, \theta_7$ are distinct (they have increasing denominators all less than 255). Also, $k/255 = -k/255 \bmod 1$ iff $k/255 = 0, 1/2 \bmod 1$. Since our points have an even denominator (except for the θ_0), we see that no point on this period-8 orbit is paired with its minus counterpart. Therefore the map $\theta_k \mapsto \sin^2(\pi\theta_k)$ is 1-1 on this periodic orbit. Since this periodic orbit has prime period 8, $x_0 = \sin^2(\frac{\pi}{255})$ is a prime-period-8 periodic point of the logistic map.

(**Q4**) Find the Poincaré map of the autonomous system

$$\ddot{x} + 2\dot{x} + 5x = 0$$

for $\Sigma = \{(x, y) = (x, 0), x > 0\}$.

Solution.

The general solution to the DE with ICs $(x(0) = a, \dot{x}(0) = 0)$ is $x(t) = ae^{-t}(\cos(2t) + \frac{1}{2}\sin(2t))$, $y(t) = \dot{x}(t) = -\frac{5}{2}ae^{-t}\sin(2t)$. Thus, $y(t) = 0$ for the first time at $t = \pi/2$ (when $x < 0$) and then at $t = \pi$ (when $x > 0$). Thus, the Poincaré map is $P(a) = x(\pi) = ae^{-\pi}$.

(**Q5**) Find the Poincaré map of the periodically forced system $\ddot{x} + x = \cos 2t$.

Solution. Here, the phase space is $\{(x, y = \dot{x}, t \bmod \pi)\} = \mathbb{R}^2 \times S^1$. The Poincaré map is from $\Sigma = \{t \equiv 0\}$ to itself. The general solution is $x(t) = (a+1/3)\cos t + b\sin t - \frac{1}{3}\cos(2t)$, $y(t) = b\cos t - (a+1/3)\sin t + \frac{2}{3}\sin(2t)$. Then $P(a,b) = (x(\pi), y(\pi)) = (-a - \frac{2}{3}, -b)$.

(**Q6**) If $\{x_n\}$ satisfies the recurrence relation

$$x_{n+1} = \lambda x_n e^{-x_n}$$

where $0 < \lambda < 1$ and $x_0 > 0$, show that $x_n \to 0$ as $n \to \infty$. Find the fixed point different from 0 which exists for $\lambda > 1$.

Solution. Let $g(x) = \lambda x e^{-x}$. If $0 < \lambda < 1$, and $x > 0$, then $0 < g(x) < x$. Therefore, the sequence x_n is monotone decreasing and bounded below by 0, hence it converges to a limit a. Since $x_{n+1} = g(x_n)$ also converges to a, we get that $a = g(a)$. Therefore, $a = \lambda a e^{-a}$ or $a = 0, a = \ln\lambda$. Since $\lambda < 1$, we see that $a = 0$. When $\lambda > 1$, we have the extra f.p. $\ln\lambda$.

(**Q7**) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Compute e^{tA}. What does this imply for the orbits of the linear system $\dot{\mathbf{x}} = A\mathbf{x}$?

Solution.

A has eigenvectors $v_+ = [1,1]'$ and $v_- = [1,-1]'$ with eigenvalues 1 and -1 respectively. Therefore

$$\begin{aligned} e^{tA} &= Pe^{t\Lambda}P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \exp(t\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}) \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \times \frac{1}{2}, \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \times \frac{1}{-2}, \\ &= \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}, \end{aligned}$$

where $\cosh t = (e^t + e^{-t})/2$ and $\sinh t = (e^t - e^{-t})/2$.

Except along the line $E^- = \mathbb{R}v_-$, all orbits will diverge to infinity. Orbits on the stable subspace E^- will converge to 0.

(**Q8**) Consider the one-dimensional map $x_{n+1} = F(x_n)$, where $F(x) = x - hx^3$.

(a) Compute F^2, the second iterate of the map.

(b) Deduce that $x = \sqrt{2/h}$ belongs to a period-2 orbit.

(c) What is the other point of this periodic orbit?

Solution.

(a) $F(F(x)) = F(x - h^3) = (x - hx^3) - h(x - hx^3)^3 = x - 2hx^3 + 3h^3x^7 + h^4x^9$.

(b) We see that $x = F(F(x))$ iff $x = 0$ or $0 = -2 + 3hx^2 - 3h^2x^4 + h^3x^6$. Substitute $u = hx^2$ gives $0 = -2 + 3u - 3u^2 + u^3$. From the constant term on guesses that $\pm1, \pm2$ may be roots, and by substitution $u = 2$ is a root. Factoring yields a quadratic with roots $\frac{1\pm\sqrt{-3}}{2}$. Therefore, the only real points x s.t. $x = F(F(x))$ are $x = 0, \pm\sqrt{\frac{2}{h}}$.

(c) Since $x = 0$ is the only fixed point of F, we must have that $F(\sqrt{\frac{2}{h}}) = -\sqrt{\frac{2}{h}}$ and $F(-\sqrt{\frac{2}{h}}) = \sqrt{\frac{2}{h}}$.

(**Q9**) Consider the system $x_{n+1} = -\frac{2}{3}x_n + y_n, \quad y_{n+1} = \frac{1}{3}(-4x_n + 5y_n)$, with $x_0 = \alpha$, $y_0 = \beta$. Show that $(x_n, y_n) \to (0,0)$ as $n \to \infty$, for any choice of α and β. Show also that the convergence is faster if $\alpha = \beta$ than if $\alpha \neq \beta$.

Solution. The eigenvectors/values of A are

$$2/3, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \qquad 1/3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The general solution to the difference equation is therefore

$$\mathbf{x}_n = a(2/3)^n \begin{bmatrix} 3 \\ 4 \end{bmatrix} + b(1/3)^n \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $\mathbf{x}_0$ has components a and b along the respective eigenvectors. Clearly $\mathbf{x}_n \to 0$ as $n \to \infty$. In general $|\mathbf{x}_n| \sim 5|a|(2/3)^n$, unless $a = 0$, when $|\mathbf{x}_n| = \sqrt{2}|b|(1/3)^n$, which converges to 0 faster.

(**Q10**) For a positive real number x let $[x]$ be the floor of x–the largest integer that is less than or equal to x – and let $\{x\} = x - [x]$ be the fractional part of x. The **Gauss map** is defined as $g(x) = \{\frac{1}{x}\}$ if $x \neq 0$ and $g(0) = 0$. Define a discrete DS

$$x_{n+1} = g(x_n).$$

(a) Show that the range of g is $[0, 1)$.

(b) Show that g has a fixed point at $x = 0$.

(c) Show that for each positive integer k there is a fixed point x_k^* of g such that $x_k^* = \frac{1}{k} - \frac{1}{k^3} + \cdots$ for $k \geq 2$. [Hint: to see the plausibility, draw the graphs of $y = g(x)$ and $y = x$.]

(d) Let a_i, $i = 1, \ldots$ be non-negative integers which satisfy the property that $a_i = 0$ implies $a_j = 0$ for all $j \geq i$. Let $\alpha = [a_1, \ldots]$ denote the continued fraction

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}}}. \tag{1}$$

It is a fact that every irrational real number $\alpha \in [0, 1)$ has a unique continued fraction expansion of this form. Assuming this fact, prove that if $x_1 = \alpha$ and $\forall n \geq 1$

$$x_{n+1} = g(x_n) \tag{2}$$

then $\forall n \geq 1$

$$a_n = [1/x_n]. \tag{3}$$

(e) Conversely, argue that if we take (2–3) to be the definition of the a_i, then $\alpha = x_1$ is equal to the right-hand side of (1). In other words, from the Gauss map, we can derive the continued fraction expansion of α.

(f) Assume that $\alpha = p/q$ is a rational number in $[0, 1]$ and write $x_n = p_n/q_n$ where p_n, q_n are coprime (by convention $0 = 0/1$). Show that while $x_n \neq 0$ the denominator q_n is monotonically decreasing: $q_{n+1} < q_n$. Show that there is some N such that $x_n = 0$ for $n \geq N$.

(g) Say that a continued fraction $[a_1, \ldots]$ has a tail of zeros if there is an N s.t. $a_n = 0$ for $n \geq N$. Prove that α is rational iff its continued fraction $[a_1, \ldots]$ has a tail of zeros iff α is eventually fixed by g.

(h) Let $\alpha \in [0, 1)$ be a prime-period-2 periodic point of the Gauss map. Show that α is the root of a quadratic polynomial with rational coefficients. What are the coefficients in terms of the continued fraction expansion of α?

(i) Say that a point $x = x_1$ is eventually periodic if there is some n such that x_n is a periodic point. For example, you showed above that all rationals are eventually periodic. Fact (Lagrange): every eventually periodic irrational number is the root of a quadratic polynomial with rational coefficients.

(h) [Maple] Compute the continued fraction expansion of $\alpha = e - 2$. The answer was known to Euler, whose 300th birthday was April 15, 2007.

Solution.

(a) Since $u \mapsto \{u\}$ maps $[1, 2)$ to $[0, 1)$ and the range of $x \mapsto 1/x$ contains $[1, 2)$, the range of g contains $[0, 1)$ so it equals $[0, 1)$.

(b) $g(0) = 0$.

(c) x is a fixed point of g iff $x = g(x)$ iff $x = 1/x - [1/x]$. Let $k = [1/x]$ be a positive integer. Then $x = 1/x - k$ or $x^2 + kx - 1 = 0$. Thus $x = \frac{-k\pm\sqrt{k^2+4}}{2} = \frac{k}{2} \times \left(-1 \pm \sqrt{1 + 4/k^2}\right)$. For the '+' case,[1] this equals $x_+ = \frac{k}{2} \times \left(\frac{2}{k^2} - \frac{2}{k^4} + O(\frac{1}{k^6})\right) = \frac{1}{k} - \frac{3}{k^3} + O(\frac{1}{k^5})$. For the '−' case, it equals $x_- = -\frac{k}{2} \times \left(2 + \frac{2}{k^2} - \frac{6}{k^4} + O(\frac{1}{k^6})\right) = -k - \frac{1}{k} + \frac{1}{k^3} + O(\frac{1}{k^5})$.

It is straightforward to verify that $k+1 > 1/x_+ > k$ so that $[1/x_+] = k$ as assumed. On the other hand, $x_- \notin [0, 1)$ so it cannot be a fixed point of g. Thus, if we let $x_k^* = x_+$ for each positive integer, then $g(x_k^*) = x_k^*$.

(d) Assume that $x_1 = \alpha$ as in (**Q10**d). If $a_2 = 0$, then $a_j = 0$ for $j \geq 2$, so $\alpha = 1/a_1$ and the claim easily follows. Otherwise, $a_2 \geq 1$ so

$$1/x_1 = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\cdots}}}}, \tag{0.4}$$

$$\Longrightarrow \qquad g(x_1) = \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\cdots}}}} \quad \text{and} \quad a_1 = [1/x_1], \tag{0.5}$$

where we used $a_2 \geq 1$ to conclude the fractional term in $1/x_1$ is less than 1.

The proof is now completed by a simple induction argument.

[1] Using the Maclaurin polynomial $\sqrt{1 + x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)$.

(e) Let $\alpha \in [0,1]$ be irrational,

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\vdots + \cfrac{1}{a_n + \cfrac{1}{\cdots}}}}}} \tag{6}$$

be the continued fraction expansion of α. We see that

$$\frac{1}{\alpha} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\vdots + \cfrac{1}{a_n + \cfrac{1}{\cdots}}}}} \tag{0.7}$$

and $a_i \in \mathbb{Z}^+$ so

$$0 < \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\vdots + \cfrac{1}{a_n + \cfrac{1}{\cdots}}}}} < 1 \tag{0.8}$$

whence, with $x_1 = \alpha$

$$a_1 = [1/\alpha] = [1/x_1] \tag{0.9}$$

$$x_2 = g(x_1) = \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\vdots + \cfrac{1}{a_n + \cfrac{1}{\cdots}}}}} \tag{0.10}$$

The proof is now completed by induction.

(f) Let $x_n = p_n/q_n$ be a rational number in $(0,1]$. The case $x_n = 1$ is trivial, so we can assume that $0 < p_n < q_n$. We can write $q_n = sp_n + r$ for unique positive integers s, r where $0 \le r < p_n$. Then

$$\left[\frac{q_n}{p_n}\right] = s \qquad\qquad \left\{\frac{q_n}{p_n}\right\} = \frac{r}{p_n} \tag{0.11}$$

whence

$$x_{n+1} = \frac{r}{p_n} = \frac{p_{n+1}}{q_{n+1}} \tag{0.12}$$

so $q_{n+1} \leq p_n < q_n$.

Thus, the sequence of denominators q_n is a strictly monotonic decreasing sequence of positive integers, so it converges to 1 in at most $N = q_1$ iterations, at which point $p_n = 0$, whence $x_n = 0$ for all $n \geq N$.

(g) Let us use part (d) to define the continued fraction expansion of any $\alpha \in [0, 1]$, where we add the convention that $[1/0] = 0$. If $\alpha = x_1$ is rational, the previous step showed that the continued fraction has an infinite tail of zeros because, by our convention, x_n is eventually zero. On the other hand, if x_n is eventually zero, the continued fraction has a tail of zeros, so then (1) shows that α is rational.

(h) Let $x \in [0, 1]$ be a period 2 point of the Gauss map; we have seen that if x is rational, then it cannot be a period 2 point unless $x = 0$, thus x is irrational. Then, we see that

$$x = x_1 = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\vdots + \cfrac{1}{a_n + \cfrac{1}{\cdots}}}}}} \tag{0.13}$$

which implies that

$$x = x_3 = \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{\vdots + \cfrac{1}{a_n + \cfrac{1}{\cdots}}}}}}, \tag{0.14}$$

by the uniqueness of the coefficients, we see

$$a_1 = a_3, a_2 = a_4, a_n = a_{n+2} \tag{0.15}$$

whence

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + x}}. \tag{0.16}$$

If we manipulate this last expression, we get

$$a_1 x^2 + a_1 a_2 x - a_2 = 0. \tag{0.17}$$

Thus, for example, the golden ratio minus one, $x = \frac{-1+\sqrt{5}}{2} \in [0, 1]$, is a root of $x^2 + x - 1 = 0$, so it has the continued fraction expansion with $a_n = 1$ for all $n \geq 1$.

(i) First, assume that $x \in [0,1]$ is a period k point for the Gauss map. By the same arguments as in the previous step, we know that x is irrational and $a_n = a_{n+k}$ for all $n \geq 1$. Therefore, we get

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\vdots + \cfrac{1}{a_k + x}}}}} \tag{0.18}$$

which implies that

$$x = \frac{rx + s}{ux + v} \tag{0.19}$$

where r, s, u, v are positive integers determined by the a_n. Thus, x is a quadratic irrational. In the general case, the coefficients $a_1, \ldots, a_K$ may be arbitrary, but then $a_n = a_{n+k}$ for all $n \geq K$. A similar argument, as above, shows that x is again a quadratic irrational.

(j)