(Q1) Consider the logistic map

 $x_{n+1} = F(x_n) = \mu x_n (1 - x_n),$ where $\mu > 0$ and $x \in [0, 1].$

- (a) Find the fixed points of this map. For which values of μ do they exist?
- (b) Find a period-2 orbit of the map (i.e., find x_0 and $x_1 \neq x_0$ such that $x_1 = F(x_0)$ and $F(x_1) = x_0$). For which values of μ does it exist?

Solution.

- (a) x is a fixed point of F iff x = F(x) iff $x = \mu x(1-x)$. Clearly x = 0 is one solution, and the other satisfies $1 = \mu(1-x)$ or $x = \frac{-1+\mu}{\mu}$. Since $0 \le x \le 1$, we require that $\mu \ge 1$ for the second fixed point to exist (when $\mu = 1$ it coincides with x = 0).
- (b) x is a period-2 periodic point iff x is a fixed point of F^2 iff x = F(F(x)). There are two obvious solutions: x = 0 and $x = \mu x(1 x)$. If we divide the polynomial $F(F(x)) x = \mu^2 x(1 x)(1 \mu x(1 x)) x$ by $\mu x(x \frac{-1 + \mu}{\mu})$ we get $\mu^2 x^2 \mu(\mu + 1)x + \mu + 1 = 0$. The quadratic root formula gives

$$x_{\pm} = \frac{\mu + 1 \pm \sqrt{(\mu - 3)(\mu + 1)}}{2\mu}$$

Clearly, these points are real only if $\mu \geq 3$ (and at $\mu = 3$, they equal $2/3 = \frac{-1+\mu}{\mu}\Big|_{\mu=3}$, which is also a period-1 periodic point). For $\mu > 3$, they are distinct from the period-1 points, so it must be the case that $F(x_{-}) = x_{+}$, $F(x_{+}) = x_{-}$.

(Q2) Show that the discrete system

$$x_{n+1} = \frac{1}{4} - \frac{1}{2a} - a^2 x_n^2$$

is equivalent to the logistic map with $\mu = a$. (Hint: the variable transformation relating the two systems is affine; i.e., $y_n = \alpha x_n + \beta$.)

Solution. Let $y = \alpha x + \beta$ or $x = \frac{y-\beta}{\alpha}$ and let $f(x) = c - a^2 x^2$ where $c = \frac{1}{4} - \frac{1}{2a}$, $g(y) = ay(1-y^2)$. We want $x_{n+1} = f(x_n)$ to imply that $y_{n+1} = g(y_n)$. This leads to

$$y_{n+1} = \alpha x_{n+1} + \beta = \alpha c + \beta - a^2 \alpha x_n^2,$$

$$= \alpha c + \beta - \frac{a^2}{\alpha^2} (y_n^2 - 2\beta y_n + \beta^2),$$

$$= \alpha c + \beta - \frac{a^2 \beta^2}{\alpha} + \frac{2a^2 \beta}{\alpha} y_n - \frac{a^2}{\alpha} y_n^2$$

and we want $y_{n+1} = ay_n(1 - y_n)$ so

$$\frac{a^2}{\alpha} = a \implies \alpha = a, \quad 2a\beta = a \implies \beta = 1/2, \quad ac + 1/2 - a/4 = 0 \implies c = \frac{1}{4} - \frac{1}{2a}$$

Thereofore the equations are consistent and y = ax + 1/2.

Remark. Let us formulate the problem in terms of maps. Let $h(x) = \alpha x + \beta$ and let h^{-1} be the inverse of h. We are seeking a solution to the equation

$$g(y) = h \circ f \circ h^{-1}(y)$$
 or $g \circ h = h \circ f$.

The first equation says that g is *conjugate* to f by the conjugacy h. Maps are conjugate iff the corresponding dynamical systems can be obtained from one another by a change of coordinates – the change of coordinates map being the conjugacy.

 $(\mathbf{Q3})$ Show that the map

 $\theta_{n+1} \equiv 2\theta_n \mod 1$

can be transformed into the logistic map with $\mu = 4$ and $0 \le x_n \le 1$ by the change of variable $x_n = \sin^2(\pi \theta_n)$. Find x_0 such that $x_8 = x_0$ but $x_1, \ldots, x_7 \ne x_0$.

Solution. 1. We want to show that $\theta_{n+1} = 2\theta_n \mod 1$ and $x_n = \sin^2(\pi\theta_n)$ implies $x_{n+1} = 4x_n(1-x_n)$. If $\theta_{n+1} = 2\theta_n \mod 1$ and $x_n = \sin^2(\pi\theta_n)$, then

$$\sin^2(\pi\theta_{n+1}) = \sin^2(2\pi\theta_n)$$

= $4\sin^2(\pi\theta_n)\cos^2(\pi\theta_n),$
= $4\sin^2(\pi\theta) (1 - \sin^2(\pi\theta_n))$
= $4x_n(1 - x_n),$
= $x_{n+1}.$

Solution 2. Let f(x) = 4x(1-x), $g(\theta) = 2\theta \mod 1$ and $h(\theta) = \sin^2(\pi\theta)$. We want to show that

$$h \circ g = f \circ h.$$

$$\begin{split} h \circ g(\theta) &= \sin^2(2\pi\theta) \\ &= 4 \sin^2(\pi\theta)(1 - \sin^2(\pi\theta)) \\ &= 4h(\theta)(1 - h(\theta)), \\ &= f \circ h(\theta). \end{split}$$

Note that the calulation comes down to basically the same thing, except that in the second case we always stay in the θ coordinate system.

To find a period-8 orbit of the logistic map, observe that if $\theta_n = \theta_0$, then $x_n = x_0$. Therefore, we see that the period-*n* orbits of the times-2 map are given by $\theta_n = 2^n \theta_0 = \theta_0 \mod 1$ or $\theta_0 = \frac{k}{2\pi-1} \mod 1$ for some integer *k*. In the case n = 0, we get $\theta_0 = \frac{k}{255} \mod 1$ or $\theta_0 = 0, 1/255, 2/255, \ldots, 254/255 \mod 1$. We see that if $\theta_0 = 1/255$, then $\theta_0, \cdots, \theta_7$ are distinct (they have increasing denominators all less than 255). Also, $k/255 = -k/255 \mod 1$ iff $k/255 = 0, 1/2 \mod 1$. Since our points have an even denominator (except for the θ_0), we see that no point on this period-8 orbit is paired with its minus counterpart. Therefore the map $\theta_k \mapsto \sin^2(\pi \theta_k)$ is 1-1 on this periodic orbit. Since this periodic orbit has prime period 8, $x_0 = \sin^2(\frac{\pi}{255})$ is a prime-period-8 periodic point of the logistic map.

(Q4) Find the Poincaré map of the autonomous system

$$\ddot{x} + 2\dot{x} + 5x = 0$$

for
$$\Sigma = \{(x, y) = (x, 0), x > 0\}.$$

Solution.

The general solution to the DE with ICs $(x(0) = a, \dot{x}(0) = 0)$ is $x(t) = ae^{-t}(\cos(2t) + \frac{1}{2}\sin(2t)), y(t) = \dot{x}(t) = -\frac{5}{2}ae^{-t}\sin(2t)$. Thus, y(t) = 0 for the first time at $t = \pi/2$ (when x < 0) and then at $t = \pi$ (when x > 0). Thus, the Poincaré map is $P(a) = x(\pi) = ae^{-\pi}$.

(Q5) Find the Poincaré map of the periodically forced system $\ddot{x} + x = \cos 2t$.

Solution. Here, the phase space is $\{(x, y = \dot{x}, t \mod \pi)\} = \mathbb{R}^2 \times S^1$. The Poincaré map is from $\Sigma = \{t \equiv 0\}$ to itself. The general solution is $x(t) = (a+1/3) \cos t + b \sin t - \frac{1}{3} \cos(2t),$ $y(t) = b \cos t - (a+1/3) \sin t + \frac{2}{3} \sin(2t)$. Then $P(a, b) = (x(\pi), y(\pi)) = (-a - \frac{2}{3}, -b)$.

(Q6) If $\{x_n\}$ satisfies the recurrence relation

$$x_{n+1} = \lambda x_n e^{-x_n}$$

where $0 < \lambda < 1$ and $x_0 > 0$, show that $x_n \to 0$ as $n \to \infty$. Find the fixed point different from 0 which exists for $\lambda > 1$.

Solution. Let $g(x) = \lambda x e^{-x}$. If $0 < \lambda < 1$, and x > 0, then 0 < g(x) < x. Therefore, the sequence x_n is monotone decreasing and bounded below by 0, hence it converges to a limit *a*. Since $x_{n+1} = g(x_n)$ also converges to *a*, we get that a = g(a). Therefore, $a = \lambda a e^{-a}$ or $a = 0, a = \ln \lambda$. Since $\lambda < 1$, we see that a = 0. When $\lambda > 1$, we have the extra f.p. $\ln \lambda$.

(Q7) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Compute e^{tA} . What does this imply for the orbits of the linear system $\dot{\mathbf{x}} = A\mathbf{x}$?

Solution.

A has eigenvectors $v_+ = [1,1]^\prime$ and $v_- = [1,-1]^\prime$ with eigenvalues 1 and -1 respectively. Therefore

$$e^{tA} = Pe^{t\Lambda}P^{-1} = \begin{bmatrix} 1 & 1\\ 1 & -1 \end{bmatrix} \exp(t\begin{bmatrix} 1 & 0\\ 0 & -1 \end{bmatrix}) \begin{bmatrix} 1 & 1\\ 1 & -1 \end{bmatrix} \times \frac{1}{2},$$
$$= \begin{bmatrix} 1 & 1\\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0\\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -1 & -1\\ -1 & 1 \end{bmatrix} \times \frac{1}{-2},$$
$$= \begin{bmatrix} \cosh t & \sinh t\\ \sinh t & \cosh t \end{bmatrix},$$

where $\cosh t = (e^t + e^{-t})/2$ and $\sinh t = (e^t - e^{-t})/2$.

Except along the line $E^- = \mathbb{R}v_-$, all orbits will diverge to infinity. Orbits on the stable subspace E^- will converge to 0.

(Q8) Consider the one-dimensional map $x_{n+1} = F(x_n)$, where $F(x) = x - hx^3$.

(a) Compute F^2 , the second iterate of the map.

- (b) Deduce that $x = \sqrt{2/h}$ belongs to a period-2 orbit.
- (c) What is the other point of this periodic orbit?

Solution.

- (a) $F(F(x)) = F(x h^3) = (x hx^3) h(x hx^3)^3 = x 2hx^3 + 3h^3x^7 + h^4x^9.$
- (b) We see that x = F(F(x)) iff x = 0 or $0 = -2 + 3hx^2 3h^2x^4 + h^3x^6$. Substitute $u = hx^2$ gives $0 = -2 + 3u 3u^2 + u^3$. From the constant term on guesses that $\pm 1, \pm 2$ may be roots, and by substitution u = 2 is a root. Factoring yields a quadratic with roots $\frac{1\pm\sqrt{-3}}{2}$. Therefore, the only real points x s.t. x = F(F(x)) are $x = 0, \pm \sqrt{\frac{2}{h}}$.

Problem Sheet 1

- (c) Since x = 0 is the only fixed point of F, we must have that $F(\sqrt{\frac{2}{h}}) = -\sqrt{\frac{2}{h}}$ and $F(-\sqrt{\frac{2}{h}}) = \sqrt{\frac{2}{h}}$.
- (Q9) Consider the system $x_{n+1} = -\frac{2}{3}x_n + y_n$, $y_{n+1} = \frac{1}{3}(-4x_n + 5y_n)$, with $x_0 = \alpha$, $y_0 = \beta$. Show that $(x_n, y_n) \to (0, 0)$ as $n \to \infty$, for any choice of α and β . Show also that the convergence is faster if $\alpha = \beta$ than if $\alpha \neq \beta$.

Solution. The eigenvectors/values of A are

$$2/3, \left[\begin{array}{c} 3\\4 \end{array}\right], \qquad 1/3, \left[\begin{array}{c} 1\\1 \end{array}\right]$$

The general solution to the difference equation is therefore

$$\mathbf{x}_n = a(2/3)^n \begin{bmatrix} 3\\4 \end{bmatrix} + b(1/3)^n \begin{bmatrix} 1\\1 \end{bmatrix},$$

where \mathbf{x}_0 has components a and b along the respective eigenvectors. Clearly $\mathbf{x}_n \to 0$ as $n \to \infty$. In general $|\mathbf{x}_n| \sim 5|a|(2/3)^n$, unless a = 0, when $|\mathbf{x}_n| = \sqrt{2}|b|(1/3)^n$, which converges to 0 faster.

(Q10) For a positive real number x let [x] be the floor of x-the largest integer that is less than or equal to x - and let $\{x\} = x - [x]$ be the fractional part of x. The **Gauss map** is defined as $g(x) = \{\frac{1}{x}\}$ if $x \neq 0$ and g(0) = 0. Define a discrete DS

$x_{n+1} = g(x_n).$

- (a) Show that the range of g is [0, 1).
- (b) Show that g has a fixed point at x = 0.
- (c) Show that for each positive integer k there is a fixed point x_k^* of g such that $x_k^* = \frac{1}{k} \frac{1}{k^3} + \cdots$ for $k \ge 2$. [Hint: to see the plausibility, draw the graphs of y = g(x) and y = x.]
- (d) Let a_i , i = 1, ... be non-negative integers which satisfy the property that $a_i = 0$ implies $a_i = 0$ for all $j \ge i$. Let $\alpha = [a_1, ...]$ denote the continued fraction

 $\alpha =$

$$\frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\dots}}}}.$$
 (1)

It is a fact that every irrational real number $\alpha \in [0, 1)$ has a unique continued fraction expansion of this form. Assuming this fact, prove that if $x_1 = \alpha$ and $\forall n \geq 1$

$$x_{n+1} = g(x_n) \tag{2}$$

then $\forall n \ge 1$

$$a_n = [1/x_n]. \tag{3}$$

(e) Conversely, argue that if we take (2–3) to be the definition of the a_i , then $\alpha = x_1$ is equal to the right-hand side of (1). In other words, from the Gauss map, we can derive the continued fraction expansion of α .

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- (f) Assume that $\alpha = p/q$ is a rational number in [0, 1] and write $x_n = p_n/q_n$ where p_n, q_n are coprime (by convention 0 = 0/1). Show that while $x_n \neq 0$ the denominator q_n is monotonically decreasing: $q_{n+1} < q_n$. Show that there is some N such that $x_n = 0$ for $n \ge N$.
- (g) Say that a continued fraction $[a_1, \ldots]$ has a tail of zeros if there is an N s.t. $a_n = 0$ for $n \ge N$. Prove that α is rational iff its continued fraction $[a_1, \ldots]$ has a tail of zeros iff α is eventually fixed by g.
- (h) Let $\alpha \in [0, 1)$ be a prime-period-2 periodic point of the Gauss map. Show that α is the root of a quadratic polynomial with rational coefficients. What are the coefficients in terms of the continued fraction expansion of α ?
- (i) Say that a point $x = x_1$ is eventually periodic if there is some *n* such that x_n is a periodic point. For example, you showed above that all rationals are eventually periodic. Fact (Lagrange): every eventually periodic irrational number is the root of a quadratic polynomial with rational coefficients.
- (h) [Maple] Compute the continued fraction expansion of $\alpha = e 2$. The answer was known to Euler, whose 300th birthday was April 15, 2007.

Solution.

- (a) Since $u \mapsto \{u\}$ maps [1,2) to [0,1) and the range of $x \mapsto 1/x$ contains [1,2), the range of g contains [0,1) so it equals [0,1).
- (b) g(0) = 0.
- (c) x is a fixed point of g iff x = g(x) iff x = 1/x [1/x]. Let k = [1/x] be a positive integer. Then x = 1/x k or $x^2 + kx 1 = 0$. Thus $x = \frac{-k \pm \sqrt{k^2 + 4}}{2} = \frac{k}{2} \times \left(-1 \pm \sqrt{1 + 4/k^2}\right)$. For the '+' case,¹ this equals $x_+ = \frac{k}{2} \times \left(\frac{2}{k^2} \frac{2}{k^4} + O(\frac{1}{k^6})\right) = \frac{1}{k} \frac{3}{k^3} + O(\frac{1}{k^5})$. For the '-' case, it equals $x_- = -\frac{k}{2} \times \left(2 + \frac{2}{k^2} \frac{6}{k^4} + O(\frac{1}{k^6})\right) = -k \frac{1}{k} + \frac{1}{k^3} + O(\frac{1}{k^5})$.

It is straightforward to verify that $k+1 > 1/x_+ > k$ so that $[1/x_+] = k$ as assumed. On the other hand, $x_- \notin [0,1)$ so it cannot be a fixed point of g. Thus, if we let $x_k^* = x_+$ for each positive integer, then $g(x_k^*) = x_k^*$.

(d) Assume that $x_1 = \alpha$ as in (Q10d). If $a_2 = 0$, then $a_j = 0$ for $j \ge 2$, so $\alpha = 1/a_1$ and the claim easily follows. Otherwise, $a_2 \ge 1$ so

$$1/x_{1} = a_{1} + \frac{1}{a_{2} + \frac{1}{a_{3} + \frac{1}{a_{4} + \frac{1}{\dots}}}},$$

$$(0.4)$$

$$\Rightarrow \quad g(x_{1}) = \frac{1}{a_{2} + \frac{1}{a_{3} + \frac{1}{a_{4} + \frac{1}{\dots}}}} \quad \text{and} \quad a_{1} = [1/x_{1}],$$

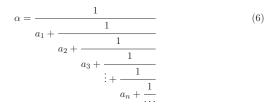
$$(0.5)$$

where we used $a_2 \ge 1$ to conclude the fractional term in $1/x_1$ is less than 1. The proof is now completed by a simple induction argument.

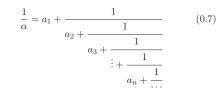
¹Using the Maclaurin polynomial $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)$.

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be the continued fraction expansion of α . We see that



and $a_i \in \mathbb{Z}^+$ so



whence, with $x_1 = \alpha$

$$a_1 = [1/\alpha] = [1/x_1] \tag{0.9}$$



The proof is now completed by induction.

(f) Let $x_n = p_n/q_n$ be a rational number in (0, 1]. The case $x_n = 1$ is trivial, so we can assume that $0 < p_n < q_n$. We can write $q_n = sp_n + r$ for unique positive integers s, r where $0 \le r < p_n$. Then

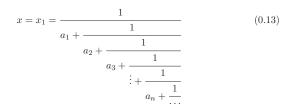
$$\begin{bmatrix} \frac{q_n}{p_n} \end{bmatrix} = s \qquad \qquad \left\{ \frac{q_n}{p_n} \right\} = \frac{r}{p_n} \qquad (0.11)$$

whence

$$x_{n+1} = \frac{r}{p_n} = \frac{p_{n+1}}{q_{n+1}} \tag{0.12}$$

so $q_{n+1} \leq p_n < q_n$.

- Thus, the sequence of denominators q_n is a strictly monotonic decreasing sequence of positive integers, so it converges to 1 in at most $N = q_1$ iterations, at which point $p_n = 0$, whence $x_n = 0$ for all $n \ge N$.
- (g) Let us use part (d) to define the continued fraction expansion of any $\alpha \in [0, 1]$, where we add the convention that [1/0] = 0. If $\alpha = x_1$ is rational, the previous step showed that the continued fraction has an infinite tail of zeros because, by our convention, x_n is eventually zero. On the other hand, if x_n is eventually zero, the continued fraction has a tail of zeros, so then (1) shows that α is rational.
- (h) Let $x \in [0,1]$ be a period 2 point of the Gauss map; we have seen that if x is rational, then it cannot be a period 2 point unless x = 0, thus x is irrational. Then, we see that



which implies that

$$x = x_3 = \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{a_5 + \frac{1}{\vdots + \frac{1}{a_n + \frac{1}{\vdots + \frac{1}{\ldots}}}}}},$$
(0.14)

by the uniqueness of the coefficients, we see

$$a_1 = a_3, a_2 = a_4, a_n = a_{n+2} \tag{0.15}$$

whence

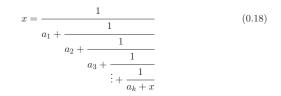
$$x = \frac{1}{a_1 + \frac{1}{a_2 + x}}.$$
(0.16)

If we manipulate this last expression, we get

$$a_1 x^2 + a_1 a_2 x - a_2 = 0. (0.17)$$

Thus, for example, the golden ratio minus one, $x = \frac{-1+\sqrt{5}}{2} \in [0,1]$, is a root of $x^2 + x - 1 = 0$, so it has the continued fraction expansion with $a_n = 1$ for all $n \ge 1$.

(i) First, assume that $x \in [0, 1]$ is a period k point for the Gauss map. By the same arguments as in the previous step, we know that x is irrational and $a_n = a_{n+k}$ for all $n \ge 1$. Therefore, we get



which implies that

$$x = \frac{rx+s}{ux+v} \tag{0.19}$$

where r, s, u, v are positive integers determined by the a_n . Thus, x is a quadratic irrational. In the general case, the coefficients a_1, \ldots, a_K may be arbitrary, but then $a_n = a_{n+k}$ for all $n \ge K$. A similar argument, as above, shows that x is again a quadratic irrational.

(j)

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