Please hand in answers no later than **Friday 26th November**.

(**Question 1**) Let $f : [0, 1] \to [0, 1]$ have the graph below. Prove that, for every $n \geq 1$, f has a prime period-n periodic orbit.

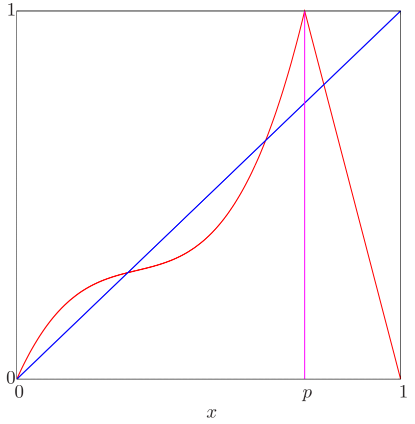

Figure .0.1: A map of $[0, 1]$ with prime periodic points of all orders.

Solution.
We shall use Sharkovskii's theorem. Recall this theorem:

Define the an ordering $\lhd$ on the positive integers by

$$3 \lhd 5 \lhd 7 \lhd \cdots \lhd 2 \cdot 3 \lhd 2 \cdot 5 \lhd 2 \cdot 7 \lhd \cdots \lhd 2^n \cdot 3 \lhd 2^n 5 \lhd 2^n 7 \lhd \cdots \cdots \lhd 2^n \lhd \cdots \lhd 2^2 \lhd 2^1 \lhd 2^0,$$

where we enumerate all odd primes in increasing order, then twice the odd primes, and so on, and finally all powers of 2.
Recall that a point x has prime period n if it is a fixed point of f^n andnot of f^k for any $k < n$.
Sharkovskii's Theorem. Let $f : [0, 1] \to [0, 1]$ be a continuous map. If f has a periodic point of prime period n, then f has a periodic point of prime period k for all $n \lhd k$.

We see that it suffices to prove that f has a prime period-3 orbit to prove that it has a prime period-n orbit for all n.

Let p be the unique point in $I = [0, 1]$ where $f(p) = 1$. Let $I_0 = [0, p]$ and $I_1 = [p, 1]$. Since f is continuous

(*) $$f(I_0) \supset I_0, I_1 \qquad f(I_1) \supset I_0, I_1.$$

We want to specify an itinerary which a period-3 point might follow. Consider the itinerary

(**) $$I_0 \to I_1 \to I_1 \to I_0.$$

Claim: If $z \in I_0$ is a period-3 point that follows the itinerary (**), then z is a prime period-3 point. Moreover, such a point z exists.

Check: It suffices to verify that z is not a fixed point of f. If $z = f(z)$, then $z \in I_0, f(z) \in I_1$ which forces $z = p$. But $f(p) = 1 \neq p$. This proves the claim.

Given (*), the intermediate value theorem (IVT) says that there is an interval $K_0 \subset I_1$ such that $f(K_0) = I_0$. Given (*), by the IVT there is an interval $K_1 \subset I_1$ such that $f(K_1) = K_0$. Finally, given (*), by the IVT there is a $K_2 \subset I_0$ such that $f(K_2) = K_1$. Then

$$f^3(K_2) = f^2(K_1) = f(K_0) = I_0 \supset K_2.$$

But therefore, as a consequence of the IVT, we know that K_2 contains a fixed point, z, of f^3. Since $z \in K_2 \subset I_0$, $f(z) \in K_1 \subset I_1$, $f^2(z) \in K_0 \subset I_1$ and $f^3(z) = z \in K_2 \subset I_0$, the period-3 point z follows the itinerary (**). Therefore z is a prime period-3 point.

(**Question 2**) Consider the $\mathbb{R}^2$ map represented as

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n) \tag{0.1}$$

where

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \mathbf{F}(\mathbf{x}) = \begin{bmatrix} y + \nu \\ y^2 - x^2 \end{bmatrix},$$

and ν is a real-valued parameter.

(a) Find the fixed points of (0.1) in terms of ν.
(b) The map (0.1) undergoes a *Hopf* bifurcation when the eigenvalues of the Jacobian matrix of derivatives of $\mathbf{F}$ are $\exp(\pm i\sigma)$, where $\sigma \in (0, \pi)$. Find the value of ν, and the corresponding fixed point, at which the map undergoes a Hopf bifurcation.

Solution.

(a) (x, y) is a fixed point of $\mathbf{F}$ iff $x = y + \nu$ and $y = y^2 - x^2$ iff $x = y + \nu$ and $y = y^2 - (y + \nu)^2 = -2\nu y - \nu^2$ iff $y = -\frac{\nu^2}{1+2\nu}$ and $x = \frac{1+\nu}{1+2\nu} \times \nu$, where $\nu \neq -1/2$.
(b) We compute that

$$d\mathbf{F}_{(x,y)} = \begin{bmatrix} 0 & 1 \\ -2x & 2y \end{bmatrix}$$

which has trace $2y$ and determinant $2x$. Its eigenvalues are therefore

$$\lambda = y \pm \sqrt{y^2 - 2x} = \frac{\nu^2 \pm \sqrt{\nu^4 - 4\nu^3 - 6\nu^2 - 2\nu}}{1 + 2\nu}.$$

Hence, if λ is properly complex, then its real part is y and imaginary part is $\pm\sqrt{2x - y^2}$ (since $y^2 - 2x < 0$). In order for λ to lie on the unit circle, we need that

$$|\lambda|^2 = 1 \iff y^2 + |y^2 - 2x| = y^2 + (2x - y^2) = 1$$
$$\iff x = 1/2, y = 1/2 - \nu.$$

To determine ν, let us solve

$$x = 1/2 \iff \nu^2 + \nu = 1/2 + \nu \iff \nu = \frac{\pm 1}{\sqrt{2}},$$

whence $y = 1/2 \mp 1/\sqrt{2}$.
Therefore, we find that there are two solutions

$$\nu = \frac{1}{\sqrt{2}}, x = \frac{1}{2}, y = x - \nu, \qquad \nu = -\frac{1}{\sqrt{2}}, x = \frac{1}{2}, y = x - \nu.$$

In the former case, $y^2 < 2x = 1$ so $\lambda = \exp(i\sigma)$, $\sigma \in (0, \pi)$ lies on the unit circle; in the latter case, $y^2 > 2x$ so λ is real. Therefore a Hopf bifurcation can only occur in the former case.