Please hand in answers no later than **Friday 19th November**.

(**Question 1**) Consider the DS $x_{n+1} = F_\nu(x_n)$ where

$$F_\nu(x_n) = \nu + x^2 ,$$

where $x \in \mathbb{R}$ and $\nu \in \mathbb{R}$.

(a) Find the fixed points. For what range of values of ν do they exist?

(b) Find the value of ν for which there is a saddle–node bifurcation.

(c) Find the value of ν for which there is a flip bifurcation. Is it super– or subcritical?

(d) Sketch the bifurcation diagram in the (ν, x) plane; indicate the stability of the fixed points in your diagram.

Solution.

(a) We want to find x s.t. $x = F_\nu(x)$. Thus $x^2 - x + \nu = 0$ so

$$x = \frac{1 \pm \sqrt{1-4\nu}}{2}.$$

Clearly, these fixed points are real iff $\nu \leq 1/4$.

(b) Let $x_\pm(\nu)$ denote the fixed points found above. Since $x_\pm(\nu) \to 1/2$ as $\nu \to 1/4$, the two fixed points merge and then disappear. Thus there is a saddle-node bifurcation at $\nu = 1/4$.

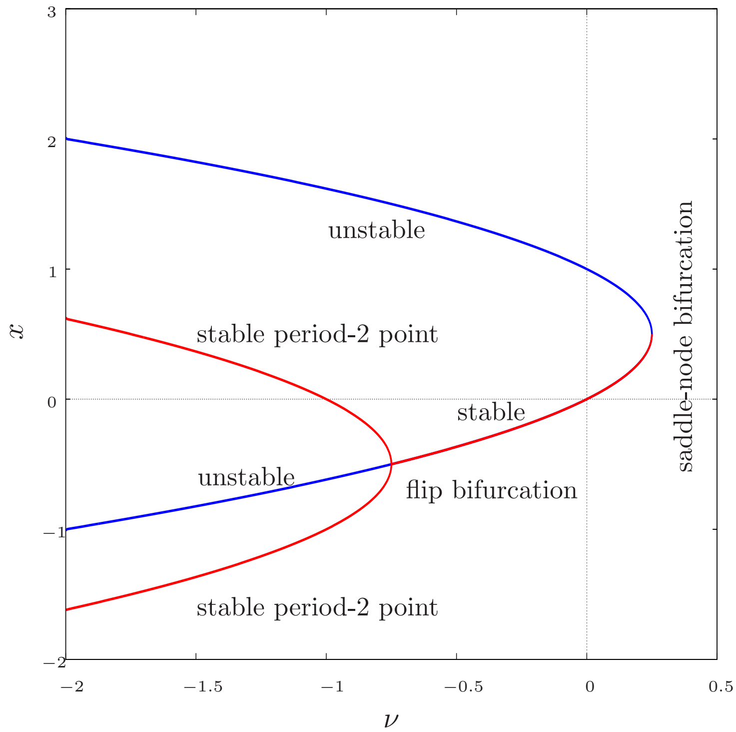
(c) There will be a flip bifurcation when $F'_\nu(x_+(\nu)) = -1$ or $F'_\nu(x_-(\nu)) = -1$. Since $F'_\mu(x) = 2x$, if the flip bifurcation occurs, then $1 - \sqrt{1-4\nu} = -1$, so $4 = 1 - 4\nu$, so $\nu = -3/4$. In this case we discarded $x_+(\nu)$ since $F'_\nu(x_+(\nu)) \geq 1$, so the flip bifurcation occurs at $x_-(\nu)$.

To determine the type of flip bifurcation, we compute the Schwartzian derivative of F_ν. Since F_ν is quadratic,

$$D_s\{F_\nu\} = \frac{F'''_\nu}{F'_\nu} - \frac{3}{2}\left(\frac{F''_\nu}{F'_\nu}\right)^2 = 0 - \frac{3}{2}\left(\frac{2}{2x}\right)^2 < 0.$$

Thus, the bifurcation at $\nu = -3/4$ is a supercritical flip bifurcation.

(d) Bifurcation diagram (in colour).



(**Question 2**) Consider the DS $x_{n+1} = H_\mu(x_n)$ with

$$H_\mu(x) = \mu \tan^{-1} x\,,$$

where x is a real variable and μ is a real parameter.

(a) How many fixed points are there? Specify the ranges of values of μ for which they exist.

(b) Calculate the Schwarzian derivative of H_μ.

(c) Describe the bifurcations which occur for

i. $\mu = 1$,

ii. $\mu = -1$.

If there are flip bifurcations, state whether they are supercritical or subcritical.

(d) Sketch the bifurcation diagram in the (μ, x) plane. Indicate the stability of the fixed points in your diagram.

Solution.

(a) For $\mu < 1$, there is exactly one fixed point $x = 0$. When $\mu > 1$, there are exactly 3 fixed points, $x = 0, \pm x(\mu)$. Let us prove this [I include this proof only because students generally did very poorly on this part].

Let $f(x) = H_\mu(x) - x$ so that $f(x) = 0$ iff x is a fixed point of H_μ. Assume that $\mu < 1$. We see that $f'(x) = \frac{\mu}{1+x^2} - 1 < \frac{1}{1+x^2} - 1 < 0$ for all $x > 0$. Therefore f is decreasing on $[0, \infty)$. Since $f(0) = 0$, $f(x) < 0$ for $x > 0$. Therefore, there are no additional fixed points in $(0, \infty)$. Since f is odd, it therefore has exactly one zero.

Let $\mu > 1$. Since $f'(0) = \mu - 1 > 0$, $f(x)$ is positive for x sufficiently small. On the other hand, since $\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$, $\lim_{x\to\infty} f(x) = -\infty$. Therefore, the intermediate value theorem says that f has a zero (call it η) in the interval $(0, \infty)$.

To see that η is the unique zero in $(0, \infty)$, assume that f has 2 zeros $0 < a < b$. Then Rolle's theorem says that, since $f(0) = f(a) = f(b) = 0$, there are c_1, c_2 s.t. $f'(c_1) = f'(c_2) = 0$ and $0 < c_1 < a < c_2 < b$. But $f'(x) = \frac{\mu}{1+x^2} - 1 = \frac{\mu-1-x^2}{1+x^2}$ which vanishes at $x = \pm\sqrt{\mu - 1}$.
Therefore: if f has more than one zero in $(0, \infty)$, then f' vanishes at least twice on $(0, \infty)$. But f' vanishes only once on this interval. Therefore f has exactly one zero in $(0, \infty)$.
By the odd symmetry of f, it has exactly 3 zeros.

Let us observe that if we use the Maclaurin series

$$\tan^{-1}(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \cdots \qquad \forall x \in (-1, 1]$$

then a non-zero fixed point x solves

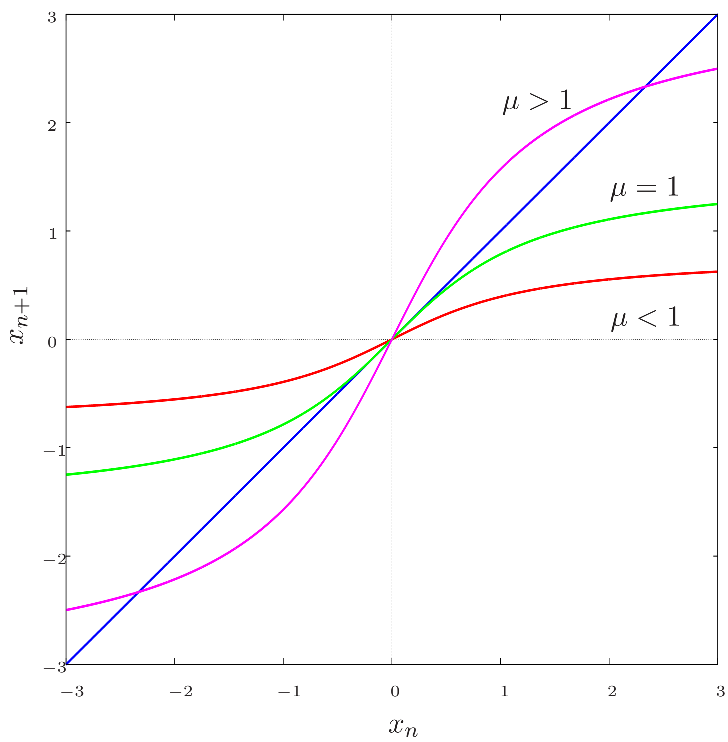
$$1 = \mu(1 - \frac{x^2}{3} + O(x^4)),$$

or

$$x = \pm\sqrt{3(\mu - 1)} + O(|\mu - 1|).$$

Or, what is better,

$$\mu = x/\tan^{-1}(x) \qquad x \neq 0.$$



(b) We compute that

$$H'_\mu = \frac{\mu}{1+x^2}, \qquad H''_\mu = \frac{-2\mu x}{(1+x^2)^2}, \qquad H'''_\mu = \frac{-2\mu}{(1+x^2)^2} + \frac{8\mu x^2}{(1+x^2)^3} = \frac{\mu(-2+6x^2)}{(1+x^2)^3}.$$

From the formula for the Schwartzian derivative, we compute that

$$D_s\{H_\mu\} = \frac{-2+6x^2}{(1+x^2)^2} - \frac{3}{2}\left(\frac{-2x}{1+x^2}\right)^2 = -\frac{2}{(1+x^2)^2}$$

(c) At $\mu = 1$, the fixed point $x = 0$ becomes unstable and gives rise to two additional stable fixed points (see the plot in i). At $\mu = -1$, we see that $H'_\mu(0) = -1$. In light of the computation of the Schwartzian derivative, we see that a supercritical flip bifurcation occurs here. That is, as μ decreases past -1, the stable fixed point $x = 0$ becomes unstable and bifurcates to produce a stable period-2 periodic orbit.

(d) Bifurcation diagram (in colour).