Please hand in answers no later than **Friday 5th November**.

(**Question 1**) Find the fixed points of the dynamical system $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n)$ where

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathbf{F}(\mathbf{x}) = \begin{pmatrix} x^2 \cos \pi y \\ y^3 \\ xz^2 + y \end{pmatrix},$$

and determine their stability.

Solution.
The fixed points satisfy $\mathbf{x} = \mathbf{F}(\mathbf{x})$ which means

$$x = x^2 \cos(\pi y), \tag{0.1}$$
$$y = y^3, \tag{0.2}$$
$$z = xz^2 + y. \tag{0.3}$$

Equation 0.2 says that $y = 0$ or $y = \pm 1$. Plugging this into equation 0.1 says that $x = x^2$ $(y = 0)$ or $x = -x^2$ $(y = \pm 1)$. Therefore $x = 0, 1$ and $y = 0$ or $x = 0, -1$ and $y = \pm 1$.

Plugging $x = 0$ into equation 0.3 yields $z = y = 0, \pm 1$. Plugging $x = 1$ and $y = 0$ into equation 0.3 yields $z^2 = z$ or $z = 0, 1$; plugging $x = -1$ and $y = \pm 1$ into equation 0.3 yields $z^2 + z - y = 0$. Since the discriminant is $1 + 4y$, this yields non-real solutions when $y = -1$ and the real solutions $z = \frac{-1 \pm \sqrt{5}}{2}$ when $y = 1$. In total, we get the following 7 real fixed points

$$x = y = z = 0; \quad x = 0, y = z = \pm 1; \quad x = 1, y = 0, z = 0, 1,$$

and

$$x = -1, y = 1, z = \frac{-1 \pm \sqrt{5}}{2}.$$

The derivative of $\mathbf{F}$ at $\mathbf{x}$ is computed to be

$$\mathbf{DF} = \begin{bmatrix} 2x\cos(\pi y) & -x^2\pi \sin(\pi y) & 0 \\ 0 & 3y^2 & 0 \\ z^2 & 1 & 2xz \end{bmatrix}$$

When $x = 0$, we see that $\mathbf{DF}$ is lower triangular with diagonal entries $0, 3y^2, 0$, so the point $(0, 0, 0)$ is stable (all eigenvalues of $\mathbf{DF}$ lie inside unit circle) while the points $(0, \pm 1, \pm 1)$ are both unstable (there is an eigenvalue outside the unit circle). When $x = 1$, $y = 0$, so $\mathbf{DF}$ is lower triangular with diagonal elements $2, 0, 2z$. Therefore $(1, 0, 0)$ and $(1, 0, 1)$ are also unstable. When $x = -1, y = 1$, then $\mathbf{DF}$ is lower triangular with diagonal entries $2, 3, -2z$ which shows it to be unstable.

In summary: only $(0, 0, 0)$ is a stable fixed point.

(**Question 2**) Consider the relation

$$x_{n+1} = x_n - x_{n-1} + 2(2x_{n-1} - x_n)^3, \tag{0.4}$$

where $x_n \in \mathbb{R}$.

(a) Introduce $y_n = x_{n-1}$ and write this relation as a dynamical system $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$ where $\mathbf{f}$ is a map from $\mathbb{R}^2$ to itself.

(b) Find the fixed points of $\mathbf{f}$.

(c) Can the corresponding linearized map determine the stability of the fixed points?

(d) Let $z_n = x_{n-1} + \epsilon x_n$ for some $\epsilon \in \mathbb{C}$. Show that, by choosing ϵ appropriately, the system (0.4) may expressed as

$$z_{n+1} = \alpha\, z_n + \beta_1\, z_n^3 + \beta_2\, z_n^2 \overline{z}_n + \beta_3\, z_n \overline{z}_n^2 + \beta_4\, \overline{z}_n^3$$

and calculate the complex-valued constants α, β_1, β_2, β_3 and β_4.

(e) Explain briefly (without performing any calculations) how the system may rewritten in terms of a new variable ζ_n as

$$\zeta_{n+1} = \alpha \zeta_n + b\, \zeta_n^2 \overline{\zeta}_n + O(|\zeta_n|^4).$$

Determine b. [NB: it is not necessary to replicate the calculations in the notes. You may use the results.]

(f) Using (iv), show how the stability of the origin depends on b/ϵ and thereby comment upon the stability of the origin for the system (0.4).

Solution.

(a) We get

$$x_{n+1} = x_n - y_n + 2(2y_n - x_n)^3,$$
$$y_{n+1} = x_n,$$

so that

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} x - y + 2(2y - x)^3 \\ x \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

(b) For a fixed point of $\mathbf{f}$ we must have $x = y$, whence $x = 2x^3$ or $x = 0$ or $\pm \frac{1}{\sqrt{2}}$.

(c) Since

$$\mathbf{Df} = \begin{bmatrix} 1 - 6(2y - x)^2 & -1 + 12(2y - x)^2 \\ 1 & 0 \end{bmatrix}$$

we see that

$$\mathbf{Df}_{x=y=0} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{Df}_{x=y=\pm\frac{1}{\sqrt{2}}} = \begin{bmatrix} -2 & 5 \\ 1 & 0 \end{bmatrix}.$$

The former has complex eigenvalues $\lambda = \lambda_\pm = \frac{1 \pm \sqrt{-3}}{2}$ which are on the unit circle (and $\lambda^3 = -1$), while the latter has determinant -5, so the second and third fixed points must be unstable.

(d) We want $z = \langle \mathbf{t}, \mathbf{x} \rangle$ where $\mathbf{t}$ is an eigenvector of $\mathbf{Df}^T$ (transpose) belonging to the eigenvalue λ. We see that $\mathbf{t} = [-\lambda, 1]^T$ is such an eigenvector.

Therefore, let $z = \langle \mathbf{t}, \mathbf{x} \rangle = y - \lambda x$ so $z_n = x_{n-1} - \lambda x_n = y_n - \lambda x_n$, where $\lambda = \frac{1 - \sqrt{-3}}{2}$. This gives the value $\epsilon = -\lambda$. Then, we see that

$$z = (y - \frac{1}{2}x) + i\frac{\sqrt{3}}{2}x, \qquad z + \bar{z} = 2y - x.$$

If we compute

$$\begin{aligned} z_{n+1} &= y_{n+1} - \lambda x_{n+1}, \\ &= x_n - \lambda(x_n - y_n + 2(2y_n - x_n)^3), \\ &= (1-\lambda)x_n + \lambda y_n - 2\lambda(z_n + \bar{z}_n)^3, \\ &= \lambda z_n - 2\lambda z_n^3 - 6\lambda z_n^2 \bar{z}_n - 6\lambda z_n \bar{z}_n^2 - 2\lambda \bar{z}_n^3. \end{aligned} \tag{0.5}$$

We have used the fact that $\lambda z = -\lambda^2 x + \lambda y = (1-\lambda)x + \lambda y$ since $\lambda^2 - \lambda + 1 = 0$.

(e) Since λ is not a cube or quartic root of unity, we would define $\zeta = h(z) = z + \alpha z^3 + \beta z^2 \bar{z} + \gamma z \bar{z}^2 + \delta \bar{z}^3$ for an appropriate choice of α, γ, δ to kill the cubic terms in the DS of equation (0.5). However, we cannot kill the $z^2 \bar{z}$ term, so we would get $\zeta_{n+1} = \lambda \zeta_n + b \zeta_n^2 \bar{\zeta}_n + O(|\zeta_n|^4)$ where b is just the coefficient on the $z^2 \bar{z}$ term in (0.5):

$$b = -6\lambda.$$

(f) We see that $|\zeta_{n+1}|^2 = |\zeta_n|^2 + 2h|\zeta_n|^3 + \cdots$ where $2h = \lambda \bar{b} + \bar{\lambda} b$ or $h = \mathrm{Re}(\bar{\lambda} b)$. Therefore, $h = -6$. This shows that the origin is a stable fixed point for (*).