Please hand in answers no later than **Tuesday 19 October**.

(**Question 1**) Consider the *linear* two-dimensional system

$$\left.\begin{aligned} x_{n+1} &= -x_n + 3y_n \\ y_{n+1} &= -\frac{3}{2}x_n + \frac{7}{2}y_n \end{aligned}\right\} \qquad (*),$$

where $x_n, y_n \in \mathbb{R}$.

(a) Show that there is a saddle-point at the origin.

(b) Find the equations of the stable and unstable subspaces at the origin.

Solution.

Put equations (*) in the form $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$ where $\mathbf{x} = [x, y]'$ and $\mathbf{A} = \begin{bmatrix} -1 & 3 \\ -3/2 & 7/2 \end{bmatrix}$. The characteristic polynomial of $\mathbf{A}$ is $\lambda^2 + 5\lambda/2 + 1$, which roots $\lambda_+ = 1/2, \lambda_- = 2$. This proves that $\mathbf{0}$ is a saddle fixed point.

The eigenspace $E^{\pm}$ is the set of $\mathbf{x}$ that solve $(\mathbf{A} - \lambda_{\pm}\mathbf{I})\mathbf{x} = \mathbf{0}$. Thus

$$E^+ = \ker(\mathbf{A} - \lambda_+) = \ker \begin{bmatrix} -1 - 1/2 & 3 \\ -3/2 & 7/2 - 1/2 \end{bmatrix} = \mathbb{R}\begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

while

$$E^- = \ker(\mathbf{A} - \lambda_-) = \ker \begin{bmatrix} -1 - 2 & 3 \\ -3/2 & 7/2 - 2 \end{bmatrix} = \mathbb{R}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Of course, E^+ is the stable subspace and E^- is the unstable subspace. For future reference, let $\mathbf{u}^{\pm}$ be the basis vector of $E^{\pm}$ specified above.

(**Question 2**) Consider the *nonlinear* two-dimensional system

$$\left.\begin{aligned} x_{n+1} &= -x_n + 3y_n - \frac{15}{8}(x_n - y_n)^3 \\ y_{n+1} &= -\frac{3}{2}x_n + \frac{7}{2}y_n - \frac{15}{8}(x_n - y_n)^3 \end{aligned}\right\}, \qquad (**)$$

where $x_n, y_n \in \mathbb{R}$.

(a) Show that there is a saddle-point at the origin.

Solution.

Put the equations (**) in the form $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n) = \mathbf{B}\mathbf{x}_n + \mathbf{G}(\mathbf{x}_n)$ where $\mathbf{x} = [x, y]'$, $\mathbf{B}$ equals the matrix $\mathbf{A}$ from the previous question and $\mathbf{G}$ is the nonlinear part. Then we see that $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ so $\mathbf{0}$ is a fixed-point, and $D\mathbf{F}_{\mathbf{0}} = \mathbf{A}$. The previous question shows that $\mathbf{0}$ is a saddle fixed point.

(b) Find the equations of the stable and unstable subspaces at the origin.

Solution.

See **Question 1**b.

(c) Introduce the vector $\begin{pmatrix} u_n^+ \\ u_n^- \end{pmatrix}$ which is defined via

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}\begin{pmatrix} u_n^+ \\ u_n^- \end{pmatrix}, \qquad (\$)$$

where $\begin{pmatrix} a \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ b \end{pmatrix}$ are vectors aligned with the stable and unstable subspaces, respectively. Thereby, show that the nonlinear system may be expressed in the form

$$\left.\begin{aligned} u_{n+1}^+ &= \alpha\, u_n^+, \\ u_{n+1}^- &= \beta\, u_n^- + \gamma\, (u_n^+)^3 \end{aligned}\right\} \qquad (\$\$)$$

and evaluate the constants a, b, α, β and γ.

Solution.

Question 1b shows that $a = 2$ and $b = 1$. [Note that ($$) has diagonalized the linear part of $\mathbf{F}$, so α and β must be the eigenvalues of $\mathbf{A}$. We show this now.] Let $\mathbf{u} = [u^+, u^-]'$ be the vector adapted to the splitting of $\mathbb{R}^2$ into E^+ and E^-. Write ($) as $\mathbf{x}_n = \mathbf{P}\mathbf{u}_n$. Then

$$\mathbf{P}\mathbf{u}_{n+1} = \mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n + \mathbf{G}(\mathbf{x}_n) = \mathbf{A}\mathbf{P}\mathbf{u}_n + \mathbf{G}(\mathbf{P}\mathbf{u}_n),$$

and so if we apply $\mathbf{P}^{-1}$ to both sides of the equation

$$\mathbf{u}_{n+1} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{u}_n + \mathbf{P}^{-1}\mathbf{G}(\mathbf{P}\mathbf{u}_n).$$

The matrix $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$. If both sides of ($) are multiplied on the left by the row vector $[1, \ -1]$, one gets $x_n - y_n = u_n^+$. Therefore

$$\mathbf{G}(\mathbf{P}\mathbf{u}_n) = \begin{bmatrix} -15/8 \cdot (u_n^+)^3 \\ -15/8 \cdot (u_n^+)^3 \end{bmatrix} = -15/8 \cdot (u_n^+)^3 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since the second column of $\mathbf{P}$ is $[1, 1]'$, we see that $\mathbf{P}\mathbf{e}_2 = [1, 1]'$ or $[0, 1]' = \mathbf{e}_2 = \mathbf{P}^{-1}[1, 1]'$. Thus

$$\mathbf{P}^{-1}\mathbf{G}(\mathbf{P}\mathbf{u}_n) = \begin{bmatrix} 0 \\ -15/8 \cdot (u_n^+)^3 \end{bmatrix}.$$

Putting this all together we get

$$\left.\begin{aligned} u_{n+1}^+ &= \tfrac{1}{2}\, u_n^+, \\ u_{n+1}^- &= 2\, u_n^- + \tfrac{-15}{8}\, (u_n^+)^3 \end{aligned}\right\}$$

or $\alpha = 1/2$, $\beta = 2$ and $\gamma = -15/8$.

(d) Show that

i. the stable manifold is given *exactly* by

$$u^- = \delta\, \left(u^+\right)^3;$$

ii. the unstable manifold is given *exactly* by

$$u^+ = \rho;$$

and evaluate the constants δ and ρ.

Solution.
The stable manifold W^+ is locally the graph of a function $f : E^+ \to E^-$ which vanishes to second order at 0. Assume $u^- = f(u^+) = a(u^+)^2 + b(u^+)^3$ exactly. Then

$$u^-_{n+1} = f(u^+_{n+1}) = f(\frac{1}{2}u^+_n) = \frac{1}{4}a(u^+_n)^2 + \frac{1}{8}b(u^+_n)^3,$$

while

$$u^-_{n+1} = 2u^-_n - \frac{15}{8}(u^+_n)^3 = 2a(u^+_n)^2 + (2b - \frac{15}{8})(u^+_n)^3.$$

Equating coefficients shows that $2a = \frac{1}{2}a$ or $a = 0$ and $2b - \frac{1}{8}b = \frac{15}{8}$ or $b = 1$. Thus

$$W^+ : \qquad (u^+, u^-) \;\; s.t. \;\; u^- = (u^+)^3.$$

The unstable manifold W^- is locally the graph of a function $g : E^- \to E^+$ which vanishes to second order at 0. You are also given that $u^+ = g(u^-) = constant$. The only possibility is that $g(u^-) = \rho$ is identically zero. Therefore $W^- = E^-$ or

$$W^- : \qquad (u^+, u^-) \;\; s.t. \;\; u^+ = 0.$$

(e) Sketch the stable and unstable manifolds in the (u^+, u^-) plane. Include in your sketch a few representative orbits and identify the stable and unstable subspaces.
Solution.

(f) Show that the nonlinear system reduces to a linear system by the variable change

$$p_n = u^+_n\,,$$
$$q_n = \delta\left(u^+_n\right)^3 - u^-_n\,.$$

Solution.
We see from ($$) that $p_{n+1} = \frac{1}{2}p_n$. On the other hand

$$q_{n+1} = (u^+_{n+1})^3 - u^-_{n+1} = \frac{1}{8}(u^+_n)^3 - 2u^-_n + \frac{15}{8}(u^+_n)^3 = 2((u^+_n)^3 - u^-_n) = 2q_n.$$

(**Question 3**) Let $I = [0, 1]$ and let $T : I \to I$ be the tent map defined by

$$T(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}] \\ -2(x-1) & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

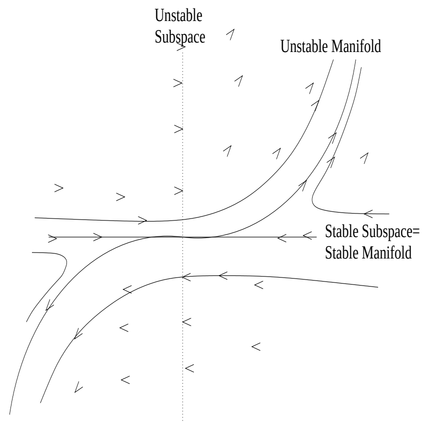

Figure .0.1:

Prove that T has exactly 2^n periodic points of period n. Compute the number of prime periodic points of period n for $n = 6$.

Solution.
Let us observe that T^n maps the interval $[k/2^n, (k+1)/2^n]$ onto $[0, 1]$ in a 1-1 fashion for each integer $k = 0, \ldots, 2^n - 1$. Indeed, on each such interval, T^n is an affine function that maps endpoints to endpoints, that is

$$T^n(x) = \begin{cases} 2^n x - k & \text{if } k \text{ is even,} \\ -2^n x + k + 1 & \text{if } k \text{ is odd.} \end{cases} \tag{0.1}$$

To prove this, note that for $n = 1$ this claim is clear, so let's assume it to be true for some $n \geq 1$. For an interval $K = [k/2^{n+1}, (k+1)/2^{n+1}]$ where $k < 2^{n+1}$, write $k = 2l + m$ where $0 \leq l < 2^n$ and $0 \leq m < 2$; the interval K is one-half of the interval $L = [l/2^n, (l+1)/2^n]$ (if $m = 0$, it is the left half; if $m = 1$, it is the right half).

Since $T^n L = [0, 1]$, and K shares an endpoint with L, the induction hypothesis implies that $T^n K \subseteq [0, 1/2]$ or $T^n K \subseteq [1/2, 1]$. Let $x \in K$, so

If $T^n x \in [0, 1/2]$,

$$T(T^n x) = \begin{cases} 2(2^n x - l) = 2^{n+1}x - 2l & \text{if } l \text{ is even,} \\ 2(-2^n x + l + 1) = -2^{n+1}x - 2(l+1) & \text{if } l \text{ is odd,} \end{cases}$$

If $T^n x \in [1/2, 1]$,

$$T(T^n x) = \begin{cases} -2(2^n x - l - 1) = -2^{n+1}x - 2(l+1) & \text{if } l \text{ is even,} \\ -2(-2^n x + l + 1 - 1) = 2^{n+1}x - 2l & \text{if } l \text{ is odd.} \end{cases}$$

It is now a matter of working through the four cases and demonstrating that $m = 0, 1, 1, 0$ in the four cases. Indeed, in the first case, $T^{n+1}x$ is monotone increasing on K, so the left end-point of K, $k/2^{n+1} = l/2^n + m/2^{n+1}$ must be mapped to 0. But since T^n maps the left end-point of L, $l/2^n$, to 0 and 0 is fixed by T, we must have that the left end-points of K and L coincide, i.e. $m = 0$ and so $k = 2l$. The other three cases are similar. Hence we have proven the claim.

Finally, since T^n is a monotone function on each of these dyadic intervals, and it maps each onto $[0, 1]$, there is a unique point in the interior of each interval that is fixed by T^n. This proves there are 2^n in total.

We have (P_n is the number of period-n points, p_n is the number of prime period-n points)

$$\begin{aligned}
&P_1 = p_1 = 2 && \\
&P_2 = 4 && p_2 = P_2 - p_1 = 2 \\
&P_3 = 8 && p_3 = P_3 - p_1 = 6 \\
&P_6 = 64 && p_6 = P_6 - p_1 - p_2 - p_3 = 54.
\end{aligned} \tag{0.2}$$