U01875 **May 2009** **MAT-4-DSy**

Dynamical Systems

(1) For each $c \in \mathbb{R}$, define a map $\mathbf{f}_c : \mathbb{R} \to \mathbb{R}$ by

$$\mathbf{f}_c(x) = c \cdot \sin(x).$$

As usual, we define a dynamical system by

$$x_{n+1} = \mathbf{f}_c(x_n) \qquad (DS)$$

for $n \geq 0$.

(a) Show that if x is a fixed point of (DS), then $-x$ is a fixed point, too. **/2**

(b) Show that $\mathbf{f}_c(\mathbb{R}) = [-|c|, |c|]$. Deduce that if x is a periodic point of $\mathbf{f}_c$, then $x \in [-|c|, |c|]$. **/3**

(c) Show that if $|c| < 1$, then for any orbit $\{x_n\}$ of (DS), x_n converges to 0. **/6**

(d) Is 0 an unstable or stable fixed point for $c \in (-1, 1)$? **/1**

(e) How many fixed points does $\mathbf{f}_c$ have for $c \in (-1, 1)$? **/1**

(f) Show that if $c > 1$, then $\mathbf{f}_c$ has at least 3 fixed points. To do this, solve for c as a function of the fixed point x and graph the resulting function. **/5**

(g) Let $c = \delta(x)$ be the function that you found in the previous question; it describes the parameter c as a function of the fixed point x. Let $\frac{\pi}{2} < \gamma < \pi$ be the smallest positive solution to the equation $x = -\tan(x)$. Determine if the 2 non-zero fixed points of $\mathbf{f}_c$ are stable or unstable for $1 < c < \delta(\gamma)$. [Remark: one can determine $\gamma \cong 2.0287578\ldots$ and $\delta(\gamma) \cong 2.2618263\ldots$.] **/4**

(h) At $c = \delta(\gamma)$, the non-zero fixed points undergo a bifurcation. Describe this bifurcation. **/3**

(2) Let $\Sigma = \mathbb{Z}_2^{\mathbb{N}} = \{(\omega_0, \omega_1, \ldots) \ : \ \omega_j \in \{0,1\} \ \forall j \geq 0\}$.

(a) Define the *shift map* $\sigma : \Sigma \to \Sigma$. **/3**

(b) Let $\omega = \overline{0110}$ be an infinite periodic sequence. Compute $\sigma^2(\omega)$. **/2**

(c) Shows that σ has exactly 2^n periodic points of period n for each $n \geq 1$. **/5**

(d) Compute the number of *prime* period n points for σ when $n = 3$ and 9. **/5**

(e) Define a metric on Σ (you do not need to prove that what you have defined is a metric). **/1**

(f) Show that σ has a dense orbit. **/4**

(g) Define sensitive dependence on initial conditions. **/2**

(h) Does σ have sensitive dependence on initial conditions? Explain. **/3**

(3) (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that f has a periodic point of prime period 3. Prove that, for all $k \geq 1$, f has a periodic point of prime period k. **/15**

(b) Let $f_\mu(x) = x + x^2 + \mu$.

(i) Find all fixed points of f_μ as a function of μ. **/3**

(ii) Describe the type of bifurcation that occurs at $\mu = 0$, if one occurs. **/2**

(c) Let $g : \mathbb{C} \to \mathbb{C}$ be defined by

$$g(z) = \left(\frac{3}{5} + i\frac{4}{5}\right) z + (2 - 3i)\, z^2 \bar{z}$$

where $i = \sqrt{-1}$. Determine the stability of the fixed point $z = 0$. **/5**

(4) Define a dynamical system on $\mathbb{R}^2$ by

$$\begin{aligned} x_{n+1} &= -\frac{16y_n}{3} + x_n^2 + \frac{17x_n}{3}, \\ y_{n+1} &= -\left(y_n + x_n\right)^2 - \frac{7y_n}{3} + \frac{8x_n}{3}. \end{aligned} \qquad (DS)$$

(a) Show that the origin is a hyperbolic fixed point of (DS). **/3**

(b) Let $\mathbf{v}_+ = \begin{bmatrix} 1 \\ * \end{bmatrix}$ (resp. $\mathbf{v}_- = \begin{bmatrix} * \\ 1 \end{bmatrix}$) span the stable (resp. unstable) subspace of $(0,0)$. Find $\mathbf{v}_+$ and $\mathbf{v}_-$. **/2**

(c) Introduce a system of coordinates (u^+, u^-) adapted to the stable and unstable subspaces. Express (DS) in the form

$$\begin{aligned} u^+_{n+1} &= au^+_n + p_0(u^+_n)^2 + p_1 u^+_n u^-_n + p_2(u^-_n)^2 \\ u^-_{n+1} &= bu^-_n + q_0(u^+_n)^2 + q_1 u^+_n u^-_n + q_2(u^-_n)^2 \end{aligned}$$

Determine the coefficients a, b, p_i, q_j for $i, j = 0, 1, 2$. **/6**

(d) Find the Maclaurin series for W^+_{loc} and W^-_{loc}, up to second order, in the coordinates (u^+, u^-). **/10**

(e) Sketch the stable and unstable subspaces and manifolds in the (u^+, u^-) coordinates. Indicate how orbits beginning on the manifolds behave and how nearby orbits behave. **/4**

[End of Paper]