U01875 **May 2009** **MAT-4-DSy**

Dynamical Systems

(1) For each $c \in \mathbb{R}$, define a map $\mathbf{f}_c : \mathbb{R} \to \mathbb{R}$ by

$$\mathbf{f}_c(x) = c \cdot \sin(x).$$

As usual, we define a dynamical system by

$$x_{n+1} = \mathbf{f}_c(x_n) \qquad (DS)$$

for $n \geq 0$.

(a) Show that if x is a fixed point of (DS), then $-x$ is a fixed point, too. **/2**

(b) Show that $\mathbf{f}_c(\mathbb{R}) = [-|c|, |c|]$. Deduce that if x is a periodic point of $\mathbf{f}_c$, then $x \in [-|c|, |c|]$. **/3**

(c) Show that if $|c| < 1$, then for any orbit $\{x_n\}$ of (DS), x_n converges to 0. **/6**

(d) Is 0 an unstable or stable fixed point for $c \in (-1, 1)$? **/1**

(e) How many fixed points does $\mathbf{f}_c$ have for $c \in (-1, 1)$? **/1**

(f) Show that if $c > 1$, then $\mathbf{f}_c$ has at least 3 fixed points. To do this, solve for c as a function of the fixed point x and graph the resulting function. **/5**

(g) Let $c = \delta(x)$ be the function that you found in the previous question; it describes the parameter c as a function of the fixed point x. Let $\frac{\pi}{2} < \gamma < \pi$ be the smallest positive solution to the equation $x = -\tan(x)$. Determine if the 2 non-zero fixed points of $\mathbf{f}_c$ are stable or unstable for $1 < c < \delta(\gamma)$. [Remark: one can determine $\gamma \cong 2.0287578\ldots$ and $\delta(\gamma) \cong 2.2618263\ldots$.] **/4**

(h) At $c = \delta(\gamma)$, the non-zero fixed points undergo a bifurcation. Describe this bifurcation. **/3**

Solution.

(a) Since $\mathbf{f}_c$ is odd: $\mathbf{f}_c(x) = x$ implies that $-x = -\mathbf{f}_c(x) = \mathbf{f}_c(-x)$ [**2 marks**].

(b) Since $-1 \leq \sin(x) \leq 1$ for all x and $\sin(x)$ attains these bounds, we have $\sin(\mathbb{R}) = [-1, 1]$, whence $\mathbf{f}_c(\mathbb{R}) = [-|c|, |c|]$ [**1 mark**]. If x is a periodic point, then there is an $n > 0$ such that $x = \mathbf{f}_c^n(x) = \mathbf{f}_c(y)$ where $y = \mathbf{f}_c^{n-1}(x)$. Thus, $x \in \mathbf{f}_c(\mathbb{R}) = [-|c|, |c|]$ [**2 marks**].

(c) For $|c| < 1$, we know that $x_1 \in (-1, 1)$. On the interval $(-1, 1)$, we know that $|\sin(x)| < |x|$, so $|\mathbf{f}_c(x)| < |c||x| < |x|$ [**2 marks**]. Therefore, $|x_1| > |x_2| > \cdots$, so $|x_n|$ is a decreasing sequence that is bounded below by 0, hence $|x_n|$ converges to some limit. Since the sign of x_n does not change, x_n converges to a limit, call it ω [**1 mark**]. Then: $\omega = \lim_{n\to\infty} x_{n+1} = \mathbf{f}_c(\lim_{n\to\infty} x_n) = \mathbf{f}_c(\omega)$, so ω is a fixed point of $\mathbf{f}_c$ in the interval $(-1, 1)$[**1 mark**]. If $\omega \neq 0$, then we have

$$\omega = \mathbf{f}_c(\omega) \quad \Longrightarrow \quad |c| = \left|\frac{\omega}{\sin(\omega)}\right| < 1 \quad \Longrightarrow \quad |\omega| < |\sin(\omega)|.$$

Absurd, since $\omega \in (-1, 1)$. Therefore $\omega = 0$. [**2 marks**]

(d) It is stable from the previous answer.

(e) Exactly one.

(f) We saw above that if x is a fixed point of $\mathbf{f}_c$, then

$$c = \delta(x) = \begin{cases} \frac{x}{\sin(x)} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0[\textbf{2 marks}]. \end{cases}$$

The function δ is even, has vertical asymptotes at $\pi\mathbb{Z}$ and it alternates in sign at each asymptote. Moreover, the minimal value of $|\delta|$ on $[k\pi, (k+1)\pi]$ is at least $|k|\pi$, so the minimum of δ is $c = 1$ attained at $x = 0$ [**2 marks**]. It follows that for $c > 1$, there are at least 3 fixed points. Here is the graph on $[-\pi, \pi]$ [**1 mark**]

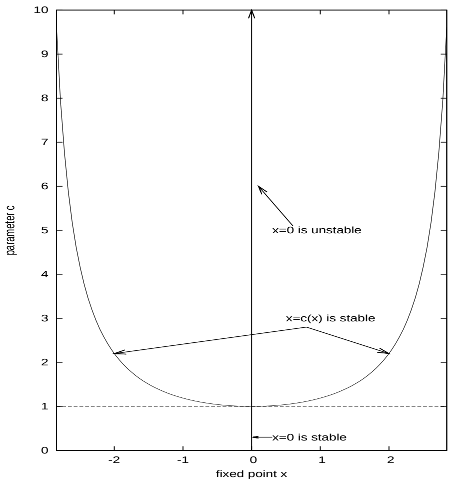

FIGURE 1

(g) Let $x > 0$ be a fixed point of $\mathbf{f}_c$ for $c = \delta(x)$. Then

$$\mathbf{f}_c'(x) = \frac{x}{\sin(x)} \times \cos(x) = \frac{x}{\tan(x)}.$$

(See figure 2.) For $0 < x < \pi/2$, $x < \tan(x)$. For $\pi/2 < x < \gamma$, $x < -\tan(x)$ and $-\tan(x)$ crosses x at $x = \gamma$. **[2 marks]**. Therefore, we see that $|\mathbf{f}_c'(x)|$ is less than 1 for $0 < x < \gamma$ (for $0 < c < \delta(\gamma)$) so the f.p. x is stable in this interval, $\mathbf{f}_c'(x)$ equals when -1 when $x = \gamma$ (i.e. $c = \delta(x)$) and $|\mathbf{f}_c'(x)|$ exceeds 1 when $x > \gamma$ so the fixed point x is unstable in this interval (i.e. $c > \delta(\gamma)$). **[2 marks]**.

(h) The above description is of a flip bifurcation. **[1 mark]**
To determine the criticality, note that when $x = \gamma$, $c = \delta(x)$

$$D_s\{\mathbf{f}_c\} = \frac{\mathbf{f}_c'''}{\mathbf{f}_c'} - \frac{3}{2}\left(\frac{\mathbf{f}_c''}{\mathbf{f}_c'}\right)^2 \leq -\mathbf{f}_c''' = \mathbf{f}_c' = -1.$$

Therefore, it is supercritical. **[2 marks]**

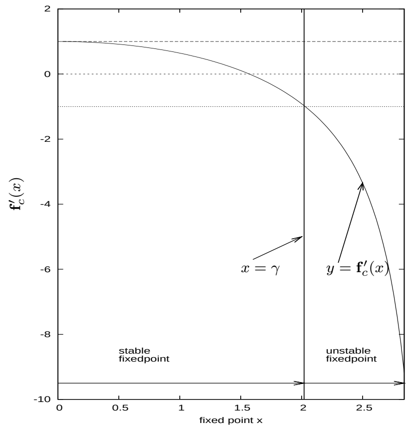

FIGURE 2

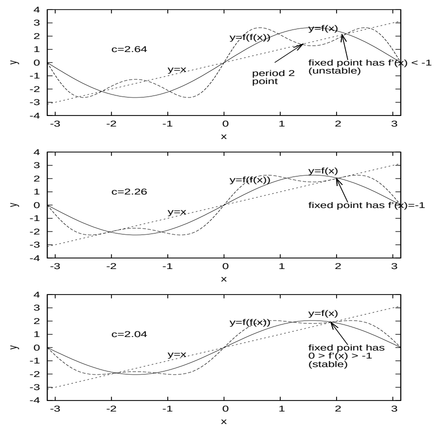

FIGURE 3

(2) Let $\Sigma = \mathbb{Z}_2^{\mathbb{N}} = \{(\omega_0, \omega_1, \ldots) \ : \ \omega_j \in \{0,1\} \ \forall j \geq 0\}$.

(a) Define the *shift map* $\sigma : \Sigma \to \Sigma$. **/3**

(b) Let $\omega = \overline{0110}$ be an infinite periodic sequence. Compute $\sigma^2(\omega)$. **/2**

(c) Shows that σ has exactly 2^n periodic points of period n for each $n \geq 1$. **/5**

(d) Compute the number of *prime* period n points for σ when $n = 3$ and 9. **/5**

(e) Define a metric on Σ (you do not need to prove that what you have defined is a metric). **/1**

(f) Show that σ has a dense orbit. **/4**

(g) Define sensitive dependence on initial conditions. **/2**

(h) Does σ have sensitive dependence on initial conditions? Explain. **/3**

Solution.

(a) For each $\omega = (\omega_0, \omega_1, \ldots) \in \Sigma$ [**1 mark**], we define

$$\sigma(\omega)_k = \omega_{k+1} \qquad \forall k \geq 0, \qquad [\textbf{2 marks}].$$

(b) We see that $\sigma^2(\omega)_k = \omega_{k+2}$ and so $\sigma^2(\overline{0110}) = \overline{1001}$ [**2 marks**].

(c) Let s be a word in $\mathbb{Z}_2$ of length n. The infinite sequence $\omega = s{\cdot}s\cdots$ (s concatenated with itself infinitely many times) lies in Σ, and $\sigma^n(\omega) = {\cdot}s\cdots = s\cdots = \omega$, so ω is a periodic point of period n. This proves there are at least 2^n periodic points of period n, since there are 2^n such words [**3 marks**].
On the other hand, $\sigma^n((\omega_0, \omega_1, \ldots)) = (\omega_n, \omega_{n+1}, \ldots)$ so ω is a fixed point iff $\omega_k = \omega_{k+n}$ for all k. Therefore, the binary word $s = \omega_0, \omega_1, \ldots, \omega_{n-1}$ determines the periodic point $\omega = s{\cdot}s\cdots$. This shows that there are at most 2^n period-n periodic points [**2 marks**].

(d) Let P_n be the number of period-n points and let p_n be the number of prime period-n points. We know that

$$p_n = P_n - \sum_{d|n, d<n} p_d, \qquad P_n = 2^n \qquad [\textbf{3 marks}].$$

Thus

$$p_1 = 2^1 \qquad p_3 = 2^3 - 2^1 = 6$$
$$p_9 = 2^9 - 6 - 2 = 504 \qquad [\textbf{2 marks}].$$

(e) Define, for all $\omega, \eta \in \Sigma$ [**1 mark**]

$$d(\omega, \eta) = \sum_{n=0}^{\infty} \frac{|\omega_n - \eta_n|}{2^{n+1}}$$

(f) Let $\omega \in \Sigma$ be constructed as follows: let s_k be the binary word obtained by concatenating all binary words of the fixed length k for $k \geq 1$. Let $\omega = s_1{\cdot}s_2\cdots$ be the concatenation of all these words [**2 marks**]. We claim that the orbit of ω is dense. Indeed, let $\eta \in \Sigma$ and $\epsilon > 0$ be given. Let N be defined to be $[\log_2 \epsilon^{-1}] + 1$. We want to show that there is an n such that $\sigma^n(\omega) \in B_\epsilon(\eta)$, or, from above, that $\sigma^n(\omega) \in C_{N+1}(\eta)$. The binary word $\eta_0, \cdots, \eta_{N+1}$ occurs in s_{N+2} and hence in ω as some subsequence $\omega_n, \cdots, \omega_{n+N+1}$ for some n. This proves that the orbit is dense since η and $\epsilon > 0$ were arbitrary [**2 marks**].

(g) We say that a map of a metric space $f : (X, d) \to (X, d)$ has s.d.i.c. if there is an $r > 0$ such that for all $x \in X$ and $\epsilon > 0$, there is a $k > 0$ and $y \in X$ such that [**2 marks**]

$$d(x, y) < \epsilon \text{ and } d(f^k x, f^k y) \geq r.$$

(h) Let d be defined as above and let $r = 1$. Let $\omega \in \Sigma$ and $\epsilon > 0$ be given. Define $N = [\log_2(\epsilon^{-1})] + 1$. We define η as follows:

$$\eta_i = \begin{cases} \omega_i & \text{if } i \leq N, \\ 1 + \omega_i & \text{if } i > N, \end{cases}$$

where addition is mod 2 [**2 marks**]. It is easy to see that $d(\omega, \eta) < \epsilon$ and $d(\sigma^N(\omega), \sigma^N(\eta)) = 1$. This proves s.d.i.c. [**1 mark**]

(3) (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that f has a periodic point of prime period 3. Prove that, for all $k \geq 1$, f has a periodic point of prime period k. **/15**

(b) Let $f_\mu(x) = x + x^2 + \mu$.

(i) Find all fixed points of f_μ as a function of μ. **/3**

(ii) Describe the type of bifurcation that occurs at $\mu = 0$, if one occurs. **/2**

(c) Let $g : \mathbb{C} \to \mathbb{C}$ be defined by

$$g(z) = \left(\frac{3}{5} + i\frac{4}{5}\right) z + (2 - 3i)\, z^2 \bar{z}$$

where $i = \sqrt{-1}$. Determine the stability of the fixed point $z = 0$. **/5**

Solution.

(a) This is textbook work (see chapter 4 of notes). To prove this, we consider the mapping F with period-3 orbit (a, b, c); i.e., we have $F(a) = b, F(b) = c, F(c) = a$. We shall assume that $a < b < c$ (the case $a < c < b$ is treated similarly) **[2 marks]**. Let us define $I_0 = [a, b]$ and $I_1 = [b, c]$ **[1 mark]**. Four observations are used in the proof **[4 marks]**:

(i) $F(I_0) \supseteq I_1$.

(ii) $F(I_1) \supseteq I_0 \cup I_1$.

(iii) If I is a closed interval and $F(I) \supseteq I$, then F has a fixed point in I.

(iv) Suppose I, J are closed intervals. If $F(I) \supseteq J$, then there exists a closed interval $K \subseteq I$ such that $F(K) = J$.

The last two observations are deduced from the intermediate value theorem, since F is continuous **[1 mark]**. We start the proof by noting that (3(a)ii) and (3(a)iii) imply that F has a fixed point in I_1 **[1 mark]**. Also, (3(a)i--3(a)iii) imply that F^2 has a fixed point in I_0, so that F has a period-2 orbit **[2 marks]**. Thus, the $n = 1$ and $n = 2$ cases are proven and henceforth we assume $n > 3$. Now we construct a nested sequence of closed intervals A_n: let $A_0 = I_1$, (3(a)ii) and (3(a)iv) imply that there is a $A_1 \subseteq A_0$ with $F(A_1) = A_0 = I_1$. Similarly, there is a $A_2 \subseteq A_1$ with $F(A_2) = A_1$ and so $F^2(A_2) = A_0$. Proceeding similarly, the sequence

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_{n-2}, \quad \text{with} \quad F^k(A_k) = A_0, \quad k = 1, 2, \ldots, n-2,$$

can be constructed **[2 marks]**. The next interval in the sequence, A_{n-1} is constructed by noting that $F^{n-1}(A_{n-2}) = F(A_0) \supseteq I_0$ (using (ii)). Then, (iv) implies that there is a $A_{n-1} \subseteq A_{n-2}$ with $F^{n-1}(A_{n-1}) = I_0$. Finally since $F^n(A_{n-1}) = F(I_0) \supseteq I_1$ (using (i)), there exists a $A_n \subseteq A_{n-1}$ with $F^n(A_n) = A_0 = I_1$. Now, by construction $A_n \subseteq A_0$, so that $F^n(A_n) \supseteq A_n$. So (iii) then implies that there exists a fixed point $x^\star \in A_n$ with $F^n(x^\star) = x^\star$. This is a prime period-n point unless it is also fixed point of F^k for $k < n$. But this is impossible since $x^\star \in A_k$, $k = 0, 1, \cdots, n$ gives that $F^k(x^\star) \in I_1$ for $k = 1, 2, \ldots, n-2$ and we also have $F^{n-1}(x^\star) \in I_0$. (The case $F^{n-1}(x^\star) \in I_0 \cap I_1 = \{b\}$ can be excluded since it would imply $n = 3$.) This completes the proof. **[2 marks]**

(b) (i) The fixed points of f_μ satisfy $x = x + x^2 + \mu$, i.e. $x = \pm\sqrt{-\mu}$ for $\mu \leq 0$ **[3 marks]**.

(ii) This is the standard example of a saddle-node (or blue-sky) bifurcation **[1 mark]**. For $\mu < 0$, there are two fixed points and these collide and disappear when $\mu = 0$ **[1 mark]**.

(c) We have that $\lambda = \frac{3+4i}{5}$ has unit modulus **[1 mark]**. Thus

$$\begin{aligned} |g(z)|^2 &= (\lambda z + cz^2\bar{z})(\bar{\lambda}\bar{z} + \bar{c}\bar{z}^2 z) \\ &= |z|^2 + (\lambda\bar{c} + \bar{\lambda}c)|z|^4 + |c|^2|z|^6 \\ &= |z|^2 + 2\mathrm{Re}(\lambda\bar{c})|z|^4 + O(|z|^6). \end{aligned}$$

[3 marks]. For z sufficiently close to 0, the $|z|^4$ term dominates the higher order terms. We compute that

$$\begin{aligned} \lambda\bar{c} &= \frac{3+4i}{5} \times (2+3i) \\ &= \frac{-6+17i}{5} \end{aligned}$$

so its real part is negative. This shows that $|g(z)| < |z|$ for all $z \neq 0$, close to 0, hence $z = 0$ is stable **[1 mark]**.

(4) Define a dynamical system on $\mathbb{R}^2$ by

$$\begin{aligned} x_{n+1} &= -\frac{16y_n}{3} + x_n^2 + \frac{17x_n}{3}, \\ y_{n+1} &= -(y_n + x_n)^2 - \frac{7y_n}{3} + \frac{8x_n}{3}. \end{aligned} \qquad (DS)$$

(a) Show that the origin is a hyperbolic fixed point of (DS). **/3**

(b) Let $\mathbf{v}_+ = \begin{bmatrix} 1 \\ * \end{bmatrix}$ (resp. $\mathbf{v}_- = \begin{bmatrix} * \\ 1 \end{bmatrix}$) span the stable (resp. unstable) subspace of $(0,0)$. Find $\mathbf{v}_+$ and $\mathbf{v}_-$. **/2**

(c) Introduce a system of coordinates (u^+, u^-) adapted to the stable and unstable subspaces. Express (DS) in the form

$$u^+_{n+1} = au^+_n + p_0(u^+_n)^2 + p_1 u^+_n u^-_n + p_2(u^-_n)^2$$
$$u^-_{n+1} = bu^-_n + q_0(u^+_n)^2 + q_1 u^+_n u^-_n + q_2(u^-_n)^2$$

Determine the coefficients a, b, p_i, q_j for $i, j = 0, 1, 2$. **/6**

(d) Find the Maclaurin series for W^+_{loc} and W^-_{loc}, up to second order, in the coordinates (u^+, u^-). **/10**

(e) Sketch the stable and unstable subspaces and manifolds in the (u^+, u^-) coordinates. Indicate how orbits beginning on the manifolds behave and how nearby orbits behave. **/4**

Solution.

(a) The linearization at $[0,0]$ has the matrix

$$\begin{bmatrix} 17/3 & -16/3 \\ 8/3 & -7/3 \end{bmatrix} \qquad [\mathbf{1\ mark}]$$

which has characteristic polynomial $x^2 - \frac{10}{3}x + 1$ and therefore its eigenvalues are $3, 1/3$ [**2 marks**].

(b) The stable eigenvector $\mathbf{v}_+$ solves

$$\begin{bmatrix} 16/3 & -16/3 \\ 8/3 & -8/3 \end{bmatrix} \times \mathbf{v}_+ = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \mathbf{v}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad [\mathbf{1\ mark}].$$

The unstable eigenvector is $\mathbf{v}_- = [2,1]^T$ by a similar computation [**1 mark**].

(c) We have that

$$\begin{bmatrix} x \\ y \end{bmatrix} = u^+\mathbf{v}_+ + u^-\mathbf{v}_- = \begin{bmatrix} u^+ + 2u^- \\ u^+ + u^- \end{bmatrix} \qquad [\mathbf{1\ mark}]$$

so,

$$\begin{bmatrix} u^- \\ u^+ \end{bmatrix} = \begin{bmatrix} x - y \\ -x + 2y \end{bmatrix} \qquad [\mathbf{1\ mark}]$$

(DS) is transformed into

$$\begin{bmatrix} u^-_{n+1} \\ u^+_{n+1} \end{bmatrix} = \begin{bmatrix} x_{n+1} - y_{n+1} \\ -x_{n+1} + 2y_{n+1} \end{bmatrix} \qquad [\mathbf{1\ mark}]$$

$$= \begin{bmatrix} -\frac{16y_n}{3} + x_n^2 + \frac{17x_n}{3} - \left(-(y_n + x_n)^2 - \frac{7y_n}{3} + \frac{8x_n}{3}\right) \\ -\left(-\frac{16y_n}{3} + x_n^2 + \frac{17x_n}{3}\right) + 2\left(-(y_n + x_n)^2 - \frac{7y_n}{3} + \frac{8x_n}{3}\right) \end{bmatrix} \qquad [\mathbf{1\ mark}]$$

$$= \begin{bmatrix} 3x_n - 3y_n + x_n^2 + (x_n + y_n)^2 \\ x_n/3 + 2y_n/3 + x_n^2 - 2(x_n + y_n)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 3u^-_n + (u^+_n + 2u^-_n)^2 + (2u^+_n + 3u^-_n)^2 \\ u^+_n/3 - (u^+_n + 2u^-_n)^2 - 2(2u^+_n + 3u^-_n)^2 \end{bmatrix}.$$

Thus,

$$\begin{aligned} &b = 3, \quad q_0 = 5, \quad q_1 = 16, \quad q_2 = 13 \\ &a = 1/3, \quad p_0 = -9, \quad p_1 = -28, \quad p_2 = -22, \end{aligned} \qquad [\mathbf{2\ marks}].$$

(d) Assume that $u^+ = g(u^-) = a_2(u^-)^2 + \cdots$ is the local unstable manifold expressed as the graph of a function up to second order [**1 mark**]. Then, if we ignore all terms in u^-_n of degree three or more,

$$\begin{aligned} u^+_{n+1} &= \frac{1}{3}u^+_n - 9(u^-_n)^2 + \cdots && \text{using part (c)} \\ &= (\frac{1}{3}a_2 - 9)(u^-_n)^2 + \cdots && \text{using } u^+_n = a_2(u^-_n)^2 + \cdots \end{aligned}$$

while,

$$\begin{aligned} u^+_{n+1} &= a_2(u^-_{n+1})^2 + \cdots && \text{using invariance} \\ &= 9a_2(u^-_n)^2 + \cdots && \text{using part (c).} \end{aligned}$$

We equate coefficients and deduce

$$a_2 = -\frac{33}{13} \qquad [\mathbf{4\ marks}].$$

Thus,

$$W^-_{loc} = \{(-33(u^-)^2/13, u^-)\} \qquad [\mathbf{1\ mark}].$$

As above, assume that $u^- = h(u^+) = b_2(u^+)^2 + \cdots$ is the local stable manifold expressed as the graph of a function up to second order [**1 mark**]. Then, if we ignore all terms in u_n^+ of degree three or more,

$$\begin{aligned} u_{n+1}^- &= 3u_n^- + 5(u_n^+)^2 + \cdots && \text{using part (c)} \\ &= (3b_2 + 5)(u_n^+)^2 + \cdots && \text{using } u_n^- = b_2(u_n^+)^2 + \cdots \end{aligned}$$

while,

$$\begin{aligned} u_{n+1}^- &= b_2(u_{n+1}^+)^2 + \cdots && \text{using invariance} \\ &= \frac{1}{9} b_2 (u_n^+)^2 + \cdots && \text{using part (c).} \end{aligned}$$

We equate coefficients and deduce

$$b_2 = -\frac{45}{26} \qquad [\mathbf{2\ \ marks}].$$

Thus,

$$W_{loc}^+ = \{(u^+, -45(u^+)^2/26)\} \qquad [\mathbf{1\ \ mark}].$$

(e)

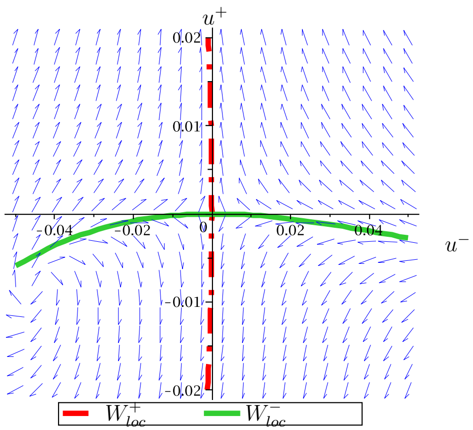

FIGURE 4. The stable and unstable manifolds of (DS). $E^{\pm} = u^{\pm}$-axis.

(i) Correct labels [**2 marks**].
(ii) Correct orientation of manifolds [**1 mark**].
(iii) Correct arrows [**1 mark**].