U01875 **May 2008** **MAT-4-DSy**

Dynamical Systems

(1) Define a map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \alpha - \beta u - v^2, \\ v \end{bmatrix}, \qquad \text{where } \mathbf{x} = \begin{bmatrix} v \\ u \end{bmatrix}$$

and $\alpha, \beta \in \mathbb{R}$ are parameters. As usual, we define a dynamical system by

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n) \qquad (DS)$$

for $n \geq 0$.

(a) Determine the set $A = \{(\alpha, \beta) \in \mathbb{R}^2 \ : \ \mathbf{f} \text{ has at least one fixed point } \}$. **/5**

(b) Determine the stability of the linearized dynamical system at each fixed point when $\alpha = 4, \beta = 2$. Are these fixed points sinks, sources, saddles or centres? **/5**

(c) When $\alpha = 0$ and $\beta = 2$, the origin $[0, 0]$ is a fixed point. Does the linearized system determine the stability of this fixed point? Explain. **/5**

(d) Continuing with $\alpha = 0, \beta = 2$, introduce the complex variable $z = \gamma u + v$ and transform (DS) into the system

$$z_{n+1} = \lambda z_n + a z_n^2 + b z_n \bar{z}_n + c \bar{z}_n^2 \qquad (CDS)$$

Determine the constants γ, λ, a, b and c. **/5**

(e) Determine the stability of the fixed point $z = 0$ for the dynamical system

$$z_{n+1} = \lambda z_n + (-3 + 4i) z_n^2 \bar{z}_n$$

where $z \in \mathbb{C}$, $\lambda = \exp\left(\frac{i\pi}{7}\right)$ and $i^2 = -1$. **/5**

(2) Let $\Sigma = \mathbb{Z}_2^{\mathbb{N}} = \{(\omega_0, \omega_1, \ldots) \ : \ \omega_j \in \{0, 1\} \ \forall j \geq 0\}$.

(a) Define the *shift map* $\sigma : \Sigma \to \Sigma$. **/5**

(b) Shows that σ has exactly 2^n periodic points of period n for each $n \geq 1$. **/5**

(c) Compute the number of *prime* period n points for σ when $n = 2, 3$ and 6. **/5**

(d) Let

$$d(\omega, \eta) = \sum_{n=0}^{\infty} \frac{|\omega_n - \eta_n|}{2^{n+1}} \qquad \forall \omega, \eta \in \Sigma.$$

You may use the fact, without proving it, that (Σ, d) is a metric space. For each $\epsilon > 0$ and $\omega \in \Sigma$, define the ball

$$B_\epsilon(\omega) = \{\eta \in \Sigma \ : \ d(\omega, \eta) < \epsilon\}$$

and, for $N \in \mathbb{N}$, the cylinder

$$C_N(\omega) = \{\eta \in \Sigma \ : \ \eta_0 = \omega_0, \cdots, \eta_N = \omega_N\}.$$

Prove: Let N be the floor of $\log_2(\epsilon^{-1}) - 1$. Then $B_\epsilon(\omega)$ is contained in $C_N(\omega)$ and $B_\epsilon(\omega)$ contains $C_{N+1}(\omega)$. **/5**

(e) Show that σ has a dense orbit. **/4**

(f) Does σ have sensitive dependence on initial conditions? Explain. **/1**

(3) Let $G(x) = 6\sin(\pi x)$ for $x \in [0,1]$.

(a) Show that there are two subintervals $I_0 = [0,a]$ and $I_1 = [b,1]$ of $I = [0,1]$ such that $G^{-1}(I) = I_0 \cup I_1$. **/2**

(b) G has two fixed points in I. Indicate their stability. **/2**

(c) Let $\Lambda = \{x \in I \ : \ \forall k \geq 0,\ G^k(x) \in I\ \}$. Describe Λ in terms of the sets I_0 and I_1. **/1**

(d) Define an itinerary map, h, for $G|\Lambda$. **/1**

(e) Show that the itinerary map is 1-1 and onto. [Indicate which, if any, theorems you use in the proof.] **/5**

(f) Show that the itinerary map h conjugates $G|\Lambda$ with the shift map $\sigma : \mathbb{Z}_2^{\mathbb{N}} \to \mathbb{Z}_2^{\mathbb{N}}$. **/3**

(g) How many period-3 points does G have? How many prime period-6 points? **/3**

(h) The map $F_\lambda(x) = -\lambda \arctan(x)$ undergoes what type of bifurcation as λ passes through 1 at $x = 0$? Explain why you know the type of bifurcation. **/5**

(i) Let $H_\mu(x) = x + x^2 - \mu$. Determine the fixed point(s) of this map in terms of μ. What type of bifurcation does this map undergo? **/3**

(4) Define a dynamical system on $\mathbb{R}^2$ by

$$\begin{aligned} x_{n+1} &= 2x_n - 4y_n + y_n^2 \\ y_{n+1} &= \tfrac{1}{2}y_n + x_n^2. \end{aligned} \qquad (DS)$$

(a) Show that the origin is a hyperbolic fixed point of (DS). **/2**

(b) Let $\mathbf{v}_+ = \begin{bmatrix} * \\ 1 \end{bmatrix}$ (resp. $\mathbf{v}_- = \begin{bmatrix} 1 \\ * \end{bmatrix}$) span the stable (resp. unstable) subspace of $(0,0)$. Find $\mathbf{v}_+$ and $\mathbf{v}_-$. **/3**

(c) Introduce a system of coordinates (u^+, u^-) adapted to the stable and unstable subspaces. Express (DS) in the form

$$u^+_{n+1} = au^+_n + p_0(u^+_n)^2 + p_1 u^+_n u^-_n + p_2(u^-_n)^2$$
$$u^-_{n+1} = bu^-_n + q_0(u^+_n)^2 + q_1 u^+_n u^-_n + q_2(u^-_n)^2$$

Determine the coefficients a, b, p_i, q_j for $i, j = 0, 1, 2$. **/6**

(d) Find the Maclaurin series for W^+_{loc} and W^-_{loc}, up to second order, in the coordinates (u^+, u^-). **/10**

(e) Sketch the stable and unstable subspaces and manifolds in the (u^+, u^-) coordinates. Indicate how orbits beginning on the manifolds behave and how nearby orbits behave. **/4**