U01875 **May 2008** **MAT-4-DSy**

Dynamical Systems

(1) Define a map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \alpha - \beta u - v^2, \\ v \end{bmatrix}, \qquad \text{where } \mathbf{x} = \begin{bmatrix} v \\ u \end{bmatrix}$$

and $\alpha, \beta \in \mathbb{R}$ are parameters. As usual, we define a dynamical system by

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n) \qquad (DS)$$

for $n \geq 0$.

(a) Determine the set $A = \{(\alpha, \beta) \in \mathbb{R}^2 \;:\; \mathbf{f}$ has at least one fixed point $\}$. **/5**

(b) Determine the stability of the linearized dynamical system at each fixed point when $\alpha = 4, \beta = 2$. Are these fixed points sinks, sources, saddles or centres? **/5**

(c) When $\alpha = 0$ and $\beta = 2$, the origin $[0, 0]$ is a fixed point. Does the linearized system determine the stability of this fixed point? Explain. **/5**

(d) Continuing with $\alpha = 0, \beta = 2$, introduce the complex variable $z = \gamma u + v$ and transform (DS) into the system

$$z_{n+1} = \lambda z_n + a z_n^2 + b z_n \bar{z}_n + c \bar{z}_n^2 \qquad (CDS)$$

Determine the constants γ, λ, a, b and c. **/5**

(e) Determine the stability of the fixed point $z = 0$ for the dynamical system

$$z_{n+1} = \lambda z_n + (-3 + 4i) z_n^2 \bar{z}_n$$

where $z \in \mathbb{C}$, $\lambda = \exp\left(\frac{i\pi}{7}\right)$ and $i^2 = -1$. **/5**

Solution.

(a) `f has a fived point at` (v, u) `iff` $v = \alpha - \beta u - v^2$ `and` $u = v$ [**2 marks**] `iff` $u = v$ `and` $v^2 + (1+\beta)v - \alpha = 0$ [**1 mark**]. `There are real f.p.s iff the discriminant`

$\Delta^2 = (1+\beta)^2 + 4\alpha$ is non-negative [**1 mark**]. Therefore,

$$\alpha \geq -\frac{1}{4}(1+\beta)^2, \quad \beta \in \mathbb{R}. \quad [\mathbf{1\ mark}]$$

(b) When $\alpha = 4, \beta = 2$, the fixed points are $u = v = 1, -4$ [**1 mark**]. The linearized map is

$$d\mathbf{f}_{[v,u]} = \begin{bmatrix} -2v & -\beta \\ 1 & 0 \end{bmatrix}. \qquad [\mathbf{1\ mark}]$$

We get

$$d\mathbf{f}_{[-4,-4]} = \begin{bmatrix} 8 & -2 \\ 1 & 0 \end{bmatrix}, \qquad d\mathbf{f}_{[1,1]} = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}. \qquad [\mathbf{1\ mark}]$$

In both cases, the determinant is $2 > 1$ so the fixed point is unstable [**1 mark**]. In fact, in the first case, the eigenvalues are $4 \pm \sqrt{14}$ so $[-4,-4]$ is a saddle; in the second case, the eigenvalues are $-1 \pm i$, so $[1,1]$ is a spiral source. [**1 mark**].

(c) The linearized map where $\alpha = 0, \beta = 2$ at $\mathbf{x} = [0,0]$ is

$$d\mathbf{f}_{[0,0]} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \qquad [\mathbf{1\ mark}]$$

which has eigenvalues $\pm i\sqrt{2}$ [**1 mark**]. The eigenvalues have modulus large than unity [**1 mark**], so the linearized system **does** determine the stability of the nonlinear system: it is an spiral source [**2 marks**].

(d) The $\lambda = i$-th eigenvector of $A = d\mathbf{f}^T_{[0,0]}$ is

$$\mathbf{t} = \begin{bmatrix} 1 \\ i\sqrt{2} \end{bmatrix}. \qquad [\mathbf{1\ mark}]$$

This gives $z = \langle \mathbf{t}, \mathbf{x} \rangle = v + i\sqrt{2}u$, $\gamma = \lambda = i$ and $\sqrt{2}u = \Im z$, $v = \Re z$. [**1 mark**].

$$\begin{aligned} z_{n+1} &= v_{n+1} + i\sqrt{2}u_{n+1}, && [\mathbf{1\ mark}] \\ &= -2u_n - v_n^2 + i\sqrt{2}v_n, \\ &= i\sqrt{2}(v_n + i\sqrt{2}u_n) - v_n^2, \\ &= i\sqrt{2}z_n - \frac{1}{4}(z_n^2 + 2z_n\bar{z}_n + \bar{z}_n^2). && [\mathbf{1\ mark}] \end{aligned}$$

This gives the result

$$\lambda \;=\; i\sqrt{2}, \quad a = c = -\tfrac{1}{4}, \quad b = -\tfrac{1}{2}. \qquad [\mathbf{1\ mark}]$$

Remark.

Many of you expected the question/solution to be of the following form:

(c') The linearized map where $\alpha = 0, \beta = 1$ at $\mathbf{x} = [0,0]$ is

$$d\mathbf{f}_{[0,0]} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad [\mathbf{1\ \ mark}]$$

which has eigenvalues $\pm i$ [**1 mark**]. The eigenvalues have unit modulus [**1 mark**], so the linearized system does not determine the stability of the nonlinear system [**2 marks**].

(d') The $\lambda = i$-th eigenvector of $A = d\mathbf{f}^T_{[0,0]}$ is

$$\mathbf{t} = \begin{bmatrix} 1 \\ i \end{bmatrix}. \qquad [\mathbf{1\ \ mark}]$$

This gives the inner product $z = \langle \mathbf{t}, \mathbf{x} \rangle = v + iu$, $\gamma = \lambda = i$ and $u = \Im z$, $v = \Re z$. [**1 mark**].

$$\begin{aligned} z_{n+1} &= v_{n+1} + iu_{n+1}, & [\mathbf{1\ \ mark}] \\ &= -u_n - v_n^2 + iv_n, \\ &= i(v_n + iu_n) - v_n^2, \\ &= iz_n - \frac{1}{4}(z_n^2 + 2z_n\bar{z}_n + \bar{z}_n^2). & [\mathbf{1\ \ mark}] \end{aligned}$$

This gives the result

$$\lambda = i, \quad a = c = -\tfrac{1}{4}, \quad b = -\tfrac{1}{2}. \qquad [\mathbf{1\ \ mark}]$$

End of Remark.

(e) We know that

$$\begin{aligned} |z_{n+1}|^2 = z_{n+1}\bar{z}_{n+1} &= (\lambda z_n + c z_n^2 \bar{z}_n)(\bar{\lambda}\bar{z}_n + \bar{c}\bar{z}_n^2 z_n) \\ &= |z_n|^2 + (c\bar{\lambda} + \bar{c}\lambda) z_n^2 \bar{z}_n^2 + |c|^2 |z_n|^6 \\ &= |z_n|^2 + 2\Re(c\bar{\lambda})|z_n|^4 + |c|^2|z_n|^6 \qquad [\mathbf{2\ \ marks}] \end{aligned}$$

where $c = -3+4i$. By the hypothesis that $\lambda = \exp\left(\frac{i\pi}{7}\right)$, we see that the real part of $c\bar{\lambda}$ is negative ($\Re(c\bar{\lambda}) \cong -0.967$) [**1 mark**]. Therefore, for small non-zero z_n, we have

$$|z_{n+1}|^2 < |z_n|^2. \qquad [\mathbf{1\ \ mark}]$$

This proves that $z = 0$ is a stable fixed point [**1 mark**].

(2) Let $\Sigma = \mathbb{Z}_2^{\mathbb{N}} = \{(\omega_0, \omega_1, \dots) \ : \ \omega_j \in \{0,1\} \ \forall j \geq 0\}$.

(a) Define the *shift map* $\sigma : \Sigma \to \Sigma$. **/5**

(b) Shows that σ has exactly 2^n periodic points of period n for each $n \geq 1$. **/5**

(c) Compute the number of *prime* period n points for σ when $n = 2, 3$ and 6. **/5**

(d) Let

$$d(\omega, \eta) = \sum_{n=0}^{\infty} \frac{|\omega_n - \eta_n|}{2^{n+1}} \qquad \forall \omega, \eta \in \Sigma.$$

You may use the fact, without proving it, that (Σ, d) is a metric space.
For each $\epsilon > 0$ and $\omega \in \Sigma$, define the ball

$$B_\epsilon(\omega) = \{\eta \in \Sigma \ : \ d(\omega, \eta) < \epsilon\}$$

and, for $N \in \mathbb{N}$, the cylinder

$$C_N(\omega) = \{\eta \in \Sigma \ : \ \eta_0 = \omega_0, \cdots, \eta_N = \omega_N\}.$$

Prove: Let N be the floor of $\log_2(\epsilon^{-1}) - 1$. Then $B_\epsilon(\omega)$ is contained in $C_N(\omega)$ and $B_\epsilon(\omega)$ contains $C_{N+1}(\omega)$. **/5**

(e) Show that σ has a dense orbit. **/4**

(f) Does σ have sensitive dependence on initial conditions? Explain. **/1**

Solution.

(a) For each $\omega = (\omega_0, \omega_1, \dots) \in \Sigma$ [**2 marks**], we define

$$\sigma(\omega)_k = \omega_{k+1} \qquad \forall k \geq 0, \qquad [\textbf{3 marks}].$$

(b) Let s be a word in $\mathbb{Z}_2$ of length n. The infinite sequence $\omega = s{\cdot}s\cdots$ (s concatenated with itself infinitely many times) lies in Σ, and $\sigma^n(\omega) = \cdot s \cdots = s \cdots = \omega$, so ω is a periodic point of period n. This proves there are at least 2^n periodic points of period n, since there are 2^n such words [**3 marks**].
On the other hand, $\sigma^n((\omega_0, \omega_1, \dots)) = (\omega_n, \omega_{n+1}, \dots)$ so ω is a fixed point iff $\omega_k = \omega_{k+n}$ for all k. Therefore,

the binary word $s = \omega_0, \omega_1, \dots, \omega_{n-1}$ determines the periodic point $\omega = s{\cdot}s\cdots$. This shows that there are at most 2^n period-n periodic points [**2 marks**].

(c) Let P_n be the number of period-n points and let p_n be the number of prime period-n points. We know that

$$p_n = P_n - \sum_{d|n, d<n} p_d, \qquad P_n = 2^n \qquad [\mathbf{3\ marks}].$$

Thus

$$p_1 = 2^1 \qquad p_2 = 2^2 - 2^1 = 2$$
$$p_3 = 2^3 - 2^1 = 6 \qquad p_6 = 2^6 - 6 - 2 - 2 = 54 \qquad [\mathbf{2\ marks}].$$

(d) Let $\epsilon > 0$ and $\omega \in \Sigma$ be given. If $\eta \in B_\epsilon(\omega)$, then

$$d(\omega, \eta) < \epsilon \qquad \Longleftrightarrow \qquad \sum_{n=0}^{\infty} \frac{|\omega_n - \eta_n|}{2^{n+1}} < \epsilon$$

which implies that for all n

$$\frac{|\omega_n - \eta_n|}{2^{n+1}} < \epsilon \qquad \Longrightarrow \qquad \omega_n = \eta_n \quad \forall n \text{ s.t. } 2^{-n-1} \geq \epsilon.$$

If we let N be the floor of $\log_2(\epsilon^{-1}) - 1$, then we arrive at

$$\eta \in B_\epsilon(\omega) \qquad \Longrightarrow \qquad \omega_0 = \eta_0, \dots, \omega_N = \eta_N.$$

Thus $\eta \in C_N(\omega)$ [**3 marks**].
On the other hand, if $\eta \in C_{N+1}(\omega)$, then

$$d(\omega, \eta) = \sum_{n=N+2} \frac{|\omega_n - \eta_n|}{2^{n+1}} \leq 2^{-N-2} \sum_{n=0} \frac{1}{2^{n+1}} = 2^{-N-2} < \epsilon.$$

Thus $\eta \in B_\epsilon(\omega)$ [**2 marks**].

(e) Let $\omega \in \Sigma$ be constructed as follows: let s_k be the binary word obtained by concatenating all binary words of the fixed length k for $k \geq 1$. Let $\omega = s_1{\cdot}s_2\cdots$ be the concatenation of all these words [**2 marks**]. We claim that the orbit of ω is dense. Indeed, let $\eta \in \Sigma$ and $\epsilon > 0$ be given. Let N be defined as in the previous question. We want to show that there is an n such that $\sigma^n(\omega) \in B_\epsilon(\eta)$, or, from above, that $\sigma^n(\omega) \in C_{N+1}(\eta)$. The binary word $\eta_0, \cdots, \eta_{N+1}$ occurs in s_{N+2} and hence in ω as some subsequence $\omega_n, \cdots, \omega_{n+N+1}$ for some n. This proves that the orbit is dense since η and $\epsilon > 0$ were arbitrary [**2 marks**].

(f) Yes, the construction of the previous question is easily adapted to prove this [**1 mark**].

(3) Let $G(x) = 6\sin(\pi x)$ for $x \in [0,1]$.

(a) Show that there are two subintervals $I_0 = [0,a]$ and $I_1 = [b,1]$ of $I = [0,1]$ such that $G^{-1}(I) = I_0 \cup I_1$. **/2**

(b) G has two fixed points in I. Indicate their stability. **/2**

(c) Let $\Lambda = \{x \in I \;:\; \forall k \geq 0,\; G^k(x) \in I\;\}$. Describe Λ in terms of the sets I_0 and I_1. **/1**

(d) Define an itinerary map, h, for $G|\Lambda$. **/1**

(e) Show that the itinerary map is 1-1 and onto. [Indicate which, if any, theorems you use in the proof.] **/5**

(f) Show that the itinerary map h conjugates $G|\Lambda$ with the shift map $\sigma : \mathbb{Z}_2^{\mathbb{N}} \to \mathbb{Z}_2^{\mathbb{N}}$. **/3**

(g) How many period-3 points does G have? How many prime period-6 points? **/3**

(h) The map $F_\lambda(x) = -\lambda \arctan(x)$ undergoes what type of bifurcation as λ passes through 1 at $x = 0$? Explain why you know the type of bifurcation. **/5**

(i) Let $H_\mu(x) = x + x^2 - \mu$. Determine the fixed point(s) of this map in terms of μ. What type of bifurcation does this map undergo? **/3**

Solution.

(a) Since G is continuous on $[0,1/2]$ and $G(0) = 0, G(1/2) = 6$, the intermediate value theorem says that there exists $a \in (0,1/2)$ s.t. $G(a) = 1$. Since G is increasing on $[0,1/2]$, a is unique. Since G is symmetric about $1/2$, the point b exists, is unique and equals $1-a$. [$a = \pi^{-1}\arcsin(1/6)$ gets only one mark.]

(b) We know that the fixed points of G lie in $I_0 \cup I_1$ since they stay in I under an iteration [**1 mark**]. We know that $G'(x) = 6\pi\cos(\pi x)$, which is decreasing on $[0,1/2]$ so $G'(x) \geq 6\pi\cos(\pi a) = 6\pi\sqrt{35}/6 > 1$ on

I_0. By symmetry, $|G'(x)| \geq \pi\sqrt{35}$ on $I_0 \cup I_1$. Therefore, the fixed points are repellers [**1 mark**].

(c) Λ is the set of points in $I_0 \cup I_1$ whose positive orbit lies in $I_0 \cup I_1$ [**1 mark**].

(d) Given $x \in \Lambda$, define the itinerary map $h(x)$ to equal $\omega \in \Sigma$ iff $G^k(x) \in I_{\omega_k}$ for all $k \geq 0$ [**1 mark**]. Since $I_0 \cap I_1 = \emptyset$, this is well-defined.

(e) Proof that h is 1-1 and onto. For each $n \geq 0$ and $\omega \in \Sigma$, define

$$I_{\omega_0,\ldots,\omega_n} = \{x \in I \ : \ G^k(x) \in I_{\omega_k} \ \forall k = 0 \ldots n \ \}. \qquad [\mathbf{1\ mark}]$$

Let $\mu = \pi\sqrt{35}$, which is a lower bound for $|G'|$ on $I_0 \cup I_1$.

CLAIM. $I_{\omega_0,\ldots,\omega_n}$ is an interval in I_{ω_0} of length $\leq \mu^{-n}$ for all ω, n [**1 mark**].

CHECK. If $n = 0$, then the claim follows since $I_{0,1}$ is an interval of length at most $1 = \mu^{-0}$. Therefore, assume the claim is true for $\leq n-1$ and all ω. The set $I_{\omega_1,\ldots,\omega_n}$ is therefore an interval in I_{ω_1} of length $\leq \mu^{-n+1}$. The set $I_{\omega_0,\ldots,\omega_n}$ is therefore the intersection of $G^{-1}(I_{\omega_1,\ldots,\omega_n})$ with I_{ω_0}. Since $G|I_{\omega_0}$ is a homeomorphism, we have proven that $I_{\omega_0,\ldots,\omega_n}$ is an interval. To prove the claim about the length, if $x, y \in I_{\omega_0,\ldots,\omega_n}$, then $|G(x) - G(y)| \leq \mu^{-n+1}$. On the other hand, the MVT plus the lower bound for $|G'|$ gives $|G(x) - G(y)| \geq \mu|x-y|$. Putting the two inequalities together shows that $|x-y| \leq \mu^{-n}$, which proves the claim [**2 marks**].

CLAIM. h is onto and 1-1.

CHECK. For each $\omega \in \Sigma$, the sets are nested: $I_{\omega_0} \supset I_{\omega_0,\omega_1} \supset \cdots I_{\omega_0,\ldots,\omega_n} \supset \cdots$. Since each is compact, their intersection is non-empty. This proves that h is onto. Since the diameter goes to zero, there is a unique point x in their intersection. This proves h is 1-1 [**1 mark**].

(f) Let $x \in \Lambda$ and let $\omega = h(x)$. Then

$$\begin{aligned} h(x) = \omega &\iff \forall k \geq 0: \quad F^k(x) \in I_{\omega_k} && [\mathbf{1\ mark}] \\ &\implies \forall k \geq 0: \quad F^k(F(x)) \in I_{\omega_{k+1}} \\ &\implies \forall k \geq 0: \quad F^k(F(x)) \in I_{\sigma(\omega)_k} && [\mathbf{1\ mark}] \\ &\implies h(F(x)) = \sigma(\omega) = \sigma(h(x)). \end{aligned}$$

Since x was arbitrary, this proves that h conjugates $F|\Lambda$ and σ [**1 mark**].

(g) Since any periodic point of G must lie in Λ, and h is a conjugacy, it suffices to count periodic points of the shift map [**2 marks**]. We have counted these already: there are 6 prime period-3 and 54 prime period-6 points [**1 mark**].

(h) We see that $F'(0) = -\lambda$, $F(0) = 0$ so as λ passes through 1 we expect a flip (period-doubling) bifurcation [**3 marks**]. Indeed, $F'(x) = -\lambda/(1+x^2)$, $F''(x) = 2x\lambda(1+x^2)^{-2}$ and $F'''(0) = 2\lambda = 2$ when $\lambda = 1$, while

$$\begin{aligned}
\left.\frac{\partial(F\circ F)'}{\partial\lambda}\right|_{\lambda=1,x=0} &= \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=1,x=0} F'(F(x))\cdot F'(x)\\
&= \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=1,x=0} \frac{\lambda^2}{(1+F^2)(1+x^2)}\\
&= \left.\frac{2\lambda}{(1+F^2)(1+x^2)} - \frac{2F^2}{(1+F^2)^2(1+x^2)}\right|_{\lambda=1,x=0}\\
&= 2.
\end{aligned}$$

This verifies the hypotheses of the period-doubling bifurcation theorem [**2 marks**].

Remark.

Many students computed the Schwartzian derivative of F at $\lambda = 1, x = 0$:

$$D_s\{F\}(0) = \frac{F'''(0)}{F'(0)} - \frac{3}{2}\left[\frac{F''(0)}{F'(0)}\right]^2 = -2.$$

They concluded that there was a supercritical flip bifurcation. This was awarded [**2 marks**], also.

End of Remark.

(i) The fixed points of H_μ satisfy $x = x + x^2 - \mu$, i.e. $x = \pm\sqrt{\mu}$ for $\mu \geq 0$ [**2 marks**]. This is the standard example of a saddle-node (or blue-sky) bifurcation [**1 mark**].

(4) Define a dynamical system on $\mathbb{R}^2$ by

$$\begin{aligned} x_{n+1} &= 2x_n - 4y_n + y_n^2 \\ y_{n+1} &= \tfrac{1}{2}y_n + x_n^2. \end{aligned} \qquad (DS)$$

(a) Show that the origin is a hyperbolic fixed point of (DS). **/2**

(b) Let $\mathbf{v}_+ = \begin{bmatrix} * \\ 1 \end{bmatrix}$ (resp. $\mathbf{v}_- = \begin{bmatrix} 1 \\ * \end{bmatrix}$) span the stable (resp. unstable) subspace of $(0,0)$. Find $\mathbf{v}_+$ and $\mathbf{v}_-$. **/3**

(c) Introduce a system of coordinates (u^+, u^-) adapted to the stable and unstable subspaces. Express (DS) in the form

$$\begin{aligned} u_{n+1}^+ &= au_n^+ + p_0(u_n^+)^2 + p_1u_n^+u_n^- + p_2(u_n^-)^2 \\ u_{n+1}^- &= bu_n^- + q_0(u_n^+)^2 + q_1u_n^+u_n^- + q_2(u_n^-)^2 \end{aligned}$$

Determine the coefficients a, b, p_i, q_j for $i, j = 0, 1, 2$. **/6**

(d) Find the Maclaurin series for W_{loc}^+ and W_{loc}^-, up to second order, in the coordinates (u^+, u^-). **/10**

(e) Sketch the stable and unstable subspaces and manifolds in the (u^+, u^-) coordinates. Indicate how orbits beginning on the manifolds behave and how nearby orbits behave. **/4**

Solution.

(a) The linearization at $[0,0]$ has the matrix

$$\begin{bmatrix} 2 & -4 \\ 0 & 1/2 \end{bmatrix} \qquad [\textbf{1 mark}]$$

which has eigenvalues $2, 1/2$ [**1 mark**].

(b) The unstable eigenvector is $\mathbf{v}_- = [1,0]^T$ [**1 mark**] while the stable eigenvector $\mathbf{v}_+$ solves

$$\begin{bmatrix} 3/2 & -4 \\ 0 & 0 \end{bmatrix} \times \mathbf{v}_+ = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \mathbf{v}_+ = \begin{bmatrix} 8/3 \\ 1 \end{bmatrix} \quad [\textbf{2 marks}].$$

(c) We have that

$$\begin{bmatrix} x \\ y \end{bmatrix} = u^+\mathbf{v}_+ + u^-\mathbf{v}_- = \begin{bmatrix} u^+8/3 + u^- \\ u^+ \end{bmatrix} \qquad [\textbf{1 mark}]$$

so,

$$\begin{bmatrix} u^- \\ u^+ \end{bmatrix} = \begin{bmatrix} x - 8y/3 \\ y \end{bmatrix} \qquad [\textbf{1 mark}]$$

(DS) is transformed into

$$\begin{aligned}
\begin{bmatrix} u^+_{n+1} \\ u^-_{n+1} \end{bmatrix} &= \begin{bmatrix} y_{n+1} \\ x_n - 8y_{n+1}/3 \end{bmatrix} && [\textbf{1 mark}] \\
&= \begin{bmatrix} y_n/2 + x_n^2 \\ 2(x_n - 8y_n/3) + y_n^2 - 8x_n^2/3 \end{bmatrix} && [\textbf{1 mark}] \\
&= \begin{bmatrix} u^+_n/2 + (u^-_n + 8u^+_n/3)^2 \\ 2u^-_n + (u^+_n)^2 - \frac{8}{3} \times (u^-_n + 8u^+_n/3)^2 \end{bmatrix} \\
&= \begin{bmatrix} u^+_n/2 + (u^-_n)^2 + 16u^-_n u^+_n/3 + 64(u^+_n)^2/9 \\ 2u^-_n - 8(u^-_n)^2/3 - 128u^-_n u^+_n/9 - 485(u^+_n)^2/27 \end{bmatrix}.
\end{aligned}$$

Thus,

$$\begin{array}{llll} a = 1/2, & p_0 = 64/9, & p_1 = 16/3, & p_2 = 1, \\ b = 2, & q_0 = -485/27, & q_1 = -128/9, & q_2 = -8/3 \end{array} \qquad [\textbf{2 marks}].$$

(d) Assume that $u^+ = g(u^-) = a_2(u^-)^2 + \cdots$ is the local unstable manifold expressed as the graph of a function up to second order [**1 mark**]. Then, if we ignore all terms in u^-_n of degree three or more,

$$\begin{aligned}
u^+_{n+1} &= \frac{1}{2}u^+_n + (u^-_n)^2 + \cdots && \text{using part (c)} \\
&= (\frac{1}{2}a_2 + 1)(u^-_n)^2 + \cdots && \text{using } u^+_n = a_2(u^-_n)^2 + \cdots
\end{aligned}$$

while,

$$\begin{aligned}
u^+_{n+1} &= a_2(u^-_{n+1})^2 + \cdots && \text{using invariance} \\
&= 4a_2(u^-_n)^2 + \cdots && \text{using part (c).}
\end{aligned}$$

We equate coefficients and deduce

$$a_2 = \frac{2}{7} \qquad [\textbf{4 marks}].$$

Thus,

$$W^-_{loc} = \{(2(u^-)^2/7, u^-)\} \qquad [\textbf{1 mark}].$$

As above, assume that $u^- = h(u^+) = b_2(u^+)^2 + \cdots$ is the local stable manifold expressed as the graph of a function up to second order [**1 mark**]. Then, if we ignore all terms in u_n^+ of degree three or more,

$$u_{n+1}^- = 2u_n^- - \frac{485}{27}(u_n^+)^2 + \cdots \qquad \text{using part (c)}$$
$$= (2b_2 - \frac{485}{27})(u_n^+)^2 + \cdots \qquad \text{using } u_n^- = b_2(u_n^+)^2 + \cdots$$

while,

$$u_{n+1}^- = b_2(u_{n+1}^+)^2 + \cdots \qquad \text{using invariance}$$
$$= \frac{1}{4}b_2(u_n^+)^2 + \cdots \qquad \text{using part (c).}$$

We equate coefficients and deduce

$$b_2 = \frac{485}{27} \times \frac{4}{7} = \frac{1940}{189} \qquad [\mathbf{2\ marks}].$$

Thus,

$$W_{loc}^+ = \{(u^+, 1940(u^+)^2/189)\} \qquad [\mathbf{1\ mark}].$$

(e)

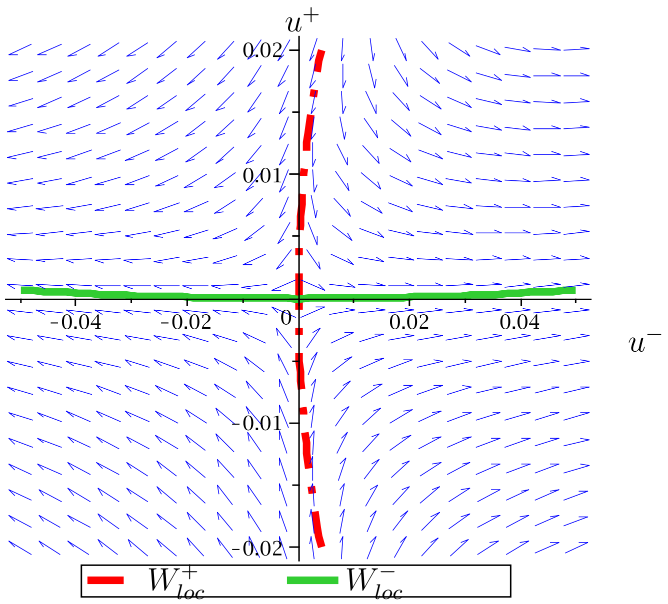

FIGURE 1. The stable and unstable manifolds of (DS). $E^{\pm} = u^{\pm}$-axis.

(i) Correct labels [**2 marks**].
(ii) Correct orientation of manifolds [**1 mark**].
(iii) Correct arrows [**1 mark**].