U01875 **May 2007** **MAT-4-DSy**

Dynamical Systems

(1) The Hénon dynamical system in $\mathbb{R}^2$ is defined by

$$\left.\begin{array}{rcl} x_{n+1} & = & a - by_n - x_n^2, \\ y_{n+1} & = & x_n \end{array}\right\} \qquad (DS),$$

where $a, b \in \mathbb{R}$ are parameters.

(a) For which range of values of the parameters a and b does (DS) have two fixed points? **/5**

(b) Determine the stability of the linearized system at each fixed point when $a = 3, b = -1$. **/5**

(c) When $a = -3/4$ and $b = 1$, the point $(-\frac{1}{2}, -\frac{1}{2})$ is a fixed point. Does the linearized system determine the stability of this fixed point? **/5**

(d) Continuing with $a = -3/4, b = 1$, introduce the coordinates $u = x + \frac{1}{2}, v = y + \frac{1}{2}$. (DS) is transformed to

$$\left.\begin{array}{rcl} u_{n+1} & = & u_n - v_n - u_n^2, \\ v_{n+1} & = & u_n \end{array}\right\} \qquad (DS'),$$

Introduce the complex variable $z = cu + v$ and transform (DS') into the system

$$z_{n+1} = \lambda z_n + \alpha z_n^2 + \beta z_n \bar{z}_n + \gamma \bar{z}_n^2 \qquad (CDS)$$

Determine the constants $c, \lambda, \alpha, \beta$ and γ. **/8**

(e) Determine the stability of the fixed point $z = 0$ for (CDS). Explain your reasoning. **/2**

A helpful formula: $h = \mathrm{Re}\left[\frac{m}{\lambda} + \frac{(2\lambda-1)\alpha\beta}{\lambda^2(\lambda-1)}\right] - \frac{1}{2}|\beta|^2 - |\gamma|^2$.

(2) Let $G(x) = 6x(1-x)$ for $x \in [0,1]$.

(a) Find the subintervals $I_0 = [0,a]$ and $I_1 = [b,1]$ of $I = [0,1]$ such that $G^{-1}(I) = I_0 \cup I_1$. **/2**

(b) G has two fixed points in I. Indicate their stability. **/3**

(c) Let $\Lambda = \{x \in I \ : \ \forall k \geq 0, \ G^k(x) \in I \ \}$. Describe Λ in terms of the sets I_0 and I_1. **/1**

(d) Let $\Sigma = \{\omega = (\omega_0, \omega_1, \ldots,) \ : \ \forall i \geq 0, \ \omega_i \in \{0,1\} \ \}$. Define a metric d on Σ. Prove that the set $U = \{\omega \in \Sigma \ : \ \omega_0 = 1, \omega_1 = 0\}$ open in the topology of (Σ, d). **/4**

(e) Define the 1-sided shift map on two symbols, $\sigma : \Sigma \to \Sigma$. **/2**

(f) Define an itinerary map, h, for $G|\Lambda$. **/1**

(g) Show that the itinerary map is continuous, 1-1 and onto. [Indicate which, if any, theorems you use in the proof.] **/7**

(h) How many period-2 points does G have? How many prime period-8 points? **/5**

(3) Define a dynamical system on $\mathbb{R}^2$ by

$$\begin{aligned} x_{n+1} &= 2x_n + 3y_n - (x_n - y_n)^2 \\ y_{n+1} &= \tfrac{1}{2}y_n + \tfrac{1}{2}(x_n - y_n)^2. \end{aligned} \qquad (DS)$$

(a) Show that the origin is a hyperbolic fixed point of (DS). **/2**

(b) Let $\mathbf{v}_+ = \begin{pmatrix} * \\ 1 \end{pmatrix}$ (resp. $\mathbf{v}_- = \begin{pmatrix} 1 \\ * \end{pmatrix}$) span the stable (resp. unstable) subspace of $(0,0)$. Find $\mathbf{v}_+$ and $\mathbf{v}_-$. **/3**

(c) Introduce a system of coordinates (u^+, u^-) adapted to the stable and unstable subspaces. Express (DS) in the form

$$\begin{aligned} u^+_{n+1} &= au^+_n + p_0(u^+_n)^2 + p_1 u^+_n u^-_n + p_2(u^-_n)^2 \\ u^-_{n+1} &= bu^-_n + q_0(u^+_n)^2 + q_1 u^+_n u^-_n + q_2(u^-_n)^2 \end{aligned} \qquad (ADS).$$

Determine the coefficients a, b, p_i, q_j for $i, j = 0, 1, 2$. **/5**

(d) Find the Maclaurin series for W^+_{loc} and W^-_{loc}, up to second order, in the coordinates (u^+, u^-). **/10**

(e) Sketch the stable and unstable subspaces and manifolds in the (u^+, u^-) coordinates. Indicate how orbits beginning on the manifolds behave, and how nearby orbits behave. **/5**

(4) (a) State Sharkovskii's theorem. **/5**

(b) Let $\mathrm{f} : [0, 1] \to [0, 1]$ be a continuous, surjective function whose graph is shown in figure 1. Prove: for each positive integer n, f has a periodic orbit of prime period n. **/8**

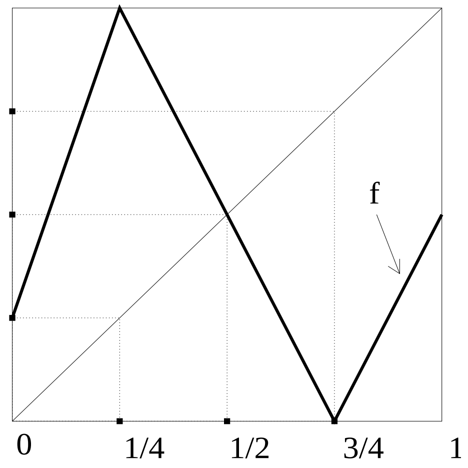

FIGURE 1. $\mathrm{f} : [0, 1] \to [0, 1]$.

(c) Let $x_{n+1} = f_\mu(x_n)$ where $f_\mu(x) = x + \mu + x^2$ for $x, \mu \in \mathbb{R}$.

(i) Find the fixed points of this dynamical system. **/2**

(ii) Find the value of μ for which there is a saddle-node bifurcation. **/1**

(iii) Find the value of μ for which there is a flip bifurcation. Is it super- or sub-critical? **/6**

(iv) Sketch the bifurcation diagram in the (μ, x) plane. **/3**

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