U01875 **May 2007** **MAT-4-DSy**

Dynamical Systems

(1) The Hénon dynamical system in $\mathbb{R}^2$ is defined by

$$\left.\begin{array}{rcl} x_{n+1} &=& a - by_n - x_n^2, \\ y_{n+1} &=& x_n \end{array}\right\} \qquad (DS),$$

where $a, b \in \mathbb{R}$ are parameters.

(a) For which range of values of the parameters a and b does (DS) have two fixed points? **/5**

(b) Determine the stability of the linearized system at each fixed point when $a = 3, b = -1$. **/5**

(c) When $a = -3/4$ and $b = 1$, the point $(-\frac{1}{2}, -\frac{1}{2})$ is a fixed point. Does the linearized system determine the stability of this fixed point? **/5**

(d) Continuing with $a = -3/4, b = 1$, introduce the coordinates $u = x + \frac{1}{2}, v = y + \frac{1}{2}$. (DS) is transformed to

$$\left.\begin{array}{rcl} u_{n+1} &=& u_n - v_n - u_n^2, \\ v_{n+1} &=& u_n \end{array}\right\} \qquad (DS'),$$

Introduce the complex variable $z = cu + v$ and transform (DS') into the system

$$z_{n+1} = \lambda z_n + \alpha z_n^2 + \beta z_n \bar{z}_n + \gamma \bar{z}_n^2 \qquad (CDS)$$

Determine the constants $c, \lambda, \alpha, \beta$ and γ. **/8**

(e) Determine the stability of the fixed point $z = 0$ for (CDS). Explain your reasoning. **/2**

A helpful formula: $h = \mathrm{Re}\left[\frac{m}{\lambda} + \frac{(2\lambda-1)\alpha\beta}{\lambda^2(\lambda-1)}\right] - \frac{1}{2}|\beta|^2 - |\gamma|^2.$

Solution.

(a) (DS) has a fixed point at (x, y) iff $x = a - by - x^2$ and $y = x$ **[2 marks]** iff $y = x$ and $x^2 + (1+b)x - a = 0$ **[1 mark]**. There are two distinct f.p.s iff the discriminant $\Delta^2 = (1+b)^2 + 4a$ is positive **[1 mark]**. Therefore,

$$a > -\frac{1}{4}(1+b)^2, \quad b \in \mathbb{R}. \quad \textbf{[1 mark]}$$

(b) When $a = 3, b = -1$, the fixed points are $x = y = \pm\sqrt{3}$ **[1 mark]**. The linearized map is

$$df_{(x,y)} = \begin{bmatrix} -2x & -b \\ 1 & 0 \end{bmatrix}. \qquad \textbf{[1 mark]}$$

We get

$$df_{(\pm\sqrt{3}, \pm\sqrt{3})} = \begin{bmatrix} \mp 2\sqrt{3} & 1 \\ 1 & 0 \end{bmatrix}. \qquad \textbf{[1 mark]}$$

The eigenvalues are $\mp(\sqrt{3} \pm 2)$ **[1 mark]**. Thus, the fixed points are saddles, hence **unstable** **[1 mark]**.

(c) The linearized map at $(-\frac{1}{2}, -\frac{1}{2})$ is

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \qquad \textbf{[1 mark]}$$

which has characteristic polynomial $\lambda^2 - \lambda + 1$ **[1 mark]**. The roots are cube roots of -1 -- hence of unit modulus **[1 mark]**. Therefore, the linearized system does not determine the stability of the nonlinear system **[2 marks]**.

(d) Let g denote the map defined by (DS'). Then $dg_{(0,0)}$ is the matrix in the previous equation **[1 mark]**. Let $\lambda, \bar{\lambda}$ be its eigenvalues with $\lambda = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. The λ-th eigenvector of $A = dg_{(0,0)}^T$ is

$$\begin{bmatrix} -\lambda \\ 1 \end{bmatrix}. \qquad \textbf{[1 mark]}$$

This gives $c = -\lambda$ **[1 mark]**. It follows that $z = (v - \frac{1}{2}u) - i\frac{\sqrt{3}}{2}u$ **[1 mark]**. Thus $u = \frac{i}{\sqrt{3}}(z - \bar{z}) = bz + \bar{b}\bar{z}$ and $v = \frac{1}{2}u + \frac{1}{2}(z + \bar{z}) = az + \bar{a}\bar{z}$ where $a = \frac{1}{2} + \frac{i}{2\sqrt{3}}$ and $b = \frac{i}{\sqrt{3}}$ **[1 mark]**. Then

$$\begin{aligned} z_{n+1} &= -\lambda u_{n+1} + v_{n+1}, \qquad \textbf{[1 mark]} \\ &= -\lambda(u_n - v_n - u_n^2) + u_n, \\ &= (1-\lambda)u_n + \lambda v_n + \lambda u_n^2, \\ &= \lambda(-\lambda)u_n + \lambda v_n + \lambda(b^2 z_n^2 + |b|^2 z\bar{z} + \bar{b}^2\bar{z}^2), \\ &= \lambda z_n + \lambda b^2 z_n^2 + 2\lambda |b|^2 z\bar{z} + \lambda\bar{b}^2\bar{z}^2, \qquad \textbf{[1 mark]} \end{aligned}$$

where we have used that $-\lambda^2 = 1-\lambda$. This proves that

$$\begin{aligned} \lambda &= \tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}, & \alpha &= \lambda b^2 = -\lambda/3, \\ \beta &= 2\lambda|b|^2 = 2\lambda/3, & \gamma &= \lambda\bar{b}^2 = -\lambda/3. \end{aligned} \quad \textbf{[1 mark]}$$

(e) We know that there is a coordinate transformation $w = f(z)$ such that our dynamical system becomes $w_{n+1} = \lambda w_n + q w_n^2 \bar{w}_n + O(|w_n|^4)$ and $|w_{n+1}|^2 = |w_n|^2 + 2h|w_n|^4 + O(|w_n|^5)$, where h is the real part of q/λ **[1 mark]**. The sign of h therefore determines the stability of 0. From the helpful formula, we know that

$$h = \mathrm{Re}\left[\frac{m}{\lambda} + \frac{(2\lambda-1)\alpha\beta}{\lambda^2(\lambda-1)}\right] - \frac{1}{2}|\beta|^2 - |\gamma|^2,$$

where $m = 0$ is the coefficient on $z_n^2\bar{z}_n$. We compute that

$$\begin{aligned} \frac{(2\lambda-1)\alpha\beta}{\lambda^2(\lambda-1)} &= \frac{i\sqrt{3}\times(-\lambda/3)\times(2\lambda/3)}{-\lambda} \\ &= \frac{i}{3\sqrt{3}} \times \left(1 + i\sqrt{3}\right) \\ &= \frac{-1}{3} + i\frac{1}{6\sqrt{3}}. \end{aligned}$$

Therefore

$$h = -\frac{1}{3} - \frac{1}{2}|\beta|^2 - |\gamma|^2 < 0.$$

Therefore, the origin is **stable** **[1 mark]**.

(2) Let $G(x) = 6x(1-x)$ for $x \in [0,1]$.

(a) Find the subintervals $I_0 = [0,a]$ and $I_1 = [b,1]$ of $I = [0,1]$ such that $G^{-1}(I) = I_0 \cup I_1$. **/2**

(b) G has two fixed points in I. Indicate their stability. **/3**

(c) Let $\Lambda = \{x \in I \ : \ \forall k \geq 0,\ G^k(x) \in I\ \}$. Describe Λ in terms of the sets I_0 and I_1. **/1**

(d) Let $\Sigma = \{\omega = (\omega_0, \omega_1, \ldots,) \ : \ \forall i \geq 0,\ \omega_i \in \{0,1\}\ \}$. Define a metric d on Σ. Prove that the set $U = \{\omega \in \Sigma \ : \omega_0 = 1, \omega_1 = 0\}$ open in the topology of (Σ, d). **/4**

(e) Define the 1-sided shift map on two symbols, $\sigma : \Sigma \to \Sigma$. **/2**

(f) Define an itinerary map, h, for $G|\Lambda$. **/1**

(g) Show that the itinerary map is continuous, 1-1 and onto. [Indicate which, if any, theorems you use in the proof.] **/7**

(h) How many period-2 points does G have? How many prime period-8 points? **/5**

Solution.

(a) We want to find solutions to $G(x) = 1$ **[1 mark]**. Thus $6x^2 - 6x + 1 = 0$ or $a = \frac{6-\sqrt{12}}{12} = \frac{1}{2} - \frac{1}{\sqrt{12}}$, $b = \frac{6+\sqrt{12}}{12} = \frac{1}{2} + \frac{1}{\sqrt{12}}$ **[1 mark]**.

(b) We know that the fixed points of G lie in $I_0 \cup I_1$ since they stay in I under an iteration **[1 mark]**. We know that $G'(x) = 6 - 12x$ so $G'(x) \geq 6 - 12a = \sqrt{12}$ on I_0 **[1 mark]**. By symmetry, $|G'(x)| \geq \sqrt{12}$ on $I_0 \cup I_1$. Therefore, the fixed points are repellers **[1 mark]**.

(c) Λ is the set of points in $I_0 \cup I_1$ whose positive orbit lies in $I_0 \cup I_1$ **[1 mark]**.

(d) We define a metric d on Σ by

$$d(\omega,\eta)=\sum_{k=0}^{\infty}\frac{|\omega_k-\eta_k|}{2^k}, \qquad [\mathbf{1\ mark}]$$

for all $\omega,\eta\in\Sigma$. To prove that U is open in (Σ,d), it suffices to prove that for all $\omega\in U$, there is a ball of radius r about ω contained in U [**1 mark**]. Now, if $d(\omega,\eta)<1/2$, then we must have that $\omega_k=\eta_k$ for $k=0,1$ [**1 mark**]. This proves that the ball of radius $1/2$ about ω is contained in U for any $\omega\in U$. Thus U is open [**1 mark**].

(e) For each $\omega\in\Sigma$: $\sigma(\omega)_k=\omega_{k+1}$ for all $k\geq 0$ [**2 marks**].

(f) Given $x\in\Lambda$, define the itinerary map $h(x)$ to equal $\omega\in\Sigma$ iff $G^k(x)\in I_{\omega_k}$ for all $k\geq 0$ [**1 mark**]. Since $I_0\cap I_1=\emptyset$, this is well-defined.

(g) Proof that h is continuous, 1-1 and onto. For each $n\geq 0$ and $\omega\in\Sigma$, define

$$I_{\omega_0,\ldots,\omega_n}=\{x\in I\ :\ G^k(x)\in I_{\omega_k}\ \forall k=0\ldots n\ \}. \qquad [\mathbf{1\ mark}]$$

Let $\mu=\sqrt{12}$, which is a lower bound for $|G'|$ on $I_0\cup I_1$.

CLAIM. $I_{\omega_0,\ldots,\omega_n}$ is an interval in I_{ω_0} of length $\leq\mu^{-n}$ for all ω,n [**1 mark**].

CHECK. If $n=0$, then the claim follows since $I_{0,1}$ is an interval of length at most $1=\mu^{-0}$. Therefore, assume the claim is true for $\leq n-1$ and all ω. The set $I_{\omega_1,\ldots,\omega_n}$ is therefore an interval in I_{ω_1} of length $\leq\mu^{-n+1}$. The set $I_{\omega_0,\ldots,\omega_n}$ is therefore the intersection of $G^{-1}(I_{\omega_1,\ldots,\omega_n})$ with I_{ω_0}. Since $G|I_{\omega_0}$ is a homeomorphism, we have proven that $I_{\omega_0,\ldots,\omega_n}$ is an interval. To prove the claim about the length, if $x,y\in I_{\omega_0,\ldots,\omega_n}$, then $|G(x)-G(y)|\leq\mu^{-n+1}$. On the other hand, the MVT plus the lower bound for $|G'|$ gives $|G(x)-G(y)|\geq\mu|x-y|$. Putting the two inequalities together shows that $|x-y|\leq\mu^{-n}$, which proves the claim [**2 marks**].

CLAIM. h is onto and 1-1.

CHECK. For each $\omega\in\Sigma$, the sets are nested: $I_{\omega_0}\supset I_{\omega_0,\omega_1}\supset\cdots I_{\omega_0,\ldots,\omega_n}\supset\cdots$. Since each is compact, their intersection in non-empty. This proves that h is onto. Since the diameter goes to zero, there is a unique point x in their intersection. This proves h is 1-1 [**1 mark**].

CLAIM. h is continuous.

CHECK. Let $x\in\Lambda$ and let $\omega=h(x)$. Let $\epsilon>0$ be

given. Choose $N>\log_2\epsilon^{-1}$ and let $\delta=6^{-N}$. Let $x'\in\Lambda$ be s.t. $|x-x'|<\delta$. Let $\omega'=h(x')$. The MVT implies that for $k<N$

$$|G^k(x)-G^k(x')|\leq 6^k|x-x'|\leq 6^{k-N}\leq 6^{-1}<b-a,$$

since $|G'|\leq 6$ on I. This implies that

$$x,x'\in\Lambda,|x-x'|<\delta \qquad\Longrightarrow\qquad d(h(x),h(x'))\leq\epsilon.$$

Indeed, if $d(\omega',\omega)>\epsilon$, then there is a smallest $k<\log_2\epsilon^{-1}<N$ s.t. $\omega'_k\neq\omega_k$. Thus, the k-th iterate of x and x' lie in opposite intervals and so they are separated by at least $b-a$. This does not happen by the above calculation. This proves the continuity of h [**2 marks**].

(h) Since any periodic point of G must lie in Λ, and h is a bijection, it suffices to count periodic points of the shift map [**2 marks**]. Let P_n (resp. p_n) be the number of period-n (resp. prime period-n) points for the shift map. We know that

$$P_n=\#\{\text{binary numbers with } n \text{ digits.}\}=2^n. \qquad [\mathbf{1\ mark}]$$

Thus

$$P_2=4.$$

On the other hand, a period-n point that is not a prime period-n point must also be a periodic point of period $k<n$, k a divisor of n [**1 mark**]. For $n=2^3$, this implies that any non-prime period-8 point is of period 4, so

$$p_8=P_8-P_4=2^8-2^4=240. \qquad [\mathbf{1\ mark}]$$

(3) Define a dynamical system on $\mathbb{R}^2$ by

$$\begin{aligned} x_{n+1} &= 2x_n + 3y_n - (x_n - y_n)^2 \\ y_{n+1} &= \tfrac{1}{2}y_n + \tfrac{1}{2}(x_n - y_n)^2. \end{aligned} \qquad (DS)$$

(a) Show that the origin is a hyperbolic fixed point of (DS). **/2**

(b) Let $\mathbf{v}_+ = \begin{pmatrix} * \\ 1 \end{pmatrix}$ (resp. $\mathbf{v}_- = \begin{pmatrix} 1 \\ * \end{pmatrix}$) span the stable (resp. unstable) subspace of $(0,0)$. Find $\mathbf{v}_+$ and $\mathbf{v}_-$. **/3**

(c) Introduce a system of coordinates (u^+, u^-) adapted to the stable and unstable subspaces. Express (DS) in the form

$$\begin{aligned} u^+_{n+1} &= au^+_n + p_0(u^+_n)^2 + p_1 u^+_n u^-_n + p_2(u^-_n)^2 \\ u^-_{n+1} &= bu^-_n + q_0(u^+_n)^2 + q_1 u^+_n u^-_n + q_2(u^-_n)^2 \end{aligned} \qquad (ADS).$$

Determine the coefficients a, b, p_i, q_j for $i, j = 0, 1, 2$. **/5**

(d) Find the Maclaurin series for W^+_{loc} and W^-_{loc}, up to second order, in the coordinates (u^+, u^-). **/10**

(e) Sketch the stable and unstable subspaces and manifolds in the (u^+, u^-) coordinates. Indicate how orbits beginning on the manifolds behave, and how nearby orbits behave. **/5**

Solution.

(a) The linearization at $(0,0)$ has the matrix

$$\begin{bmatrix} 2 & 3 \\ 0 & 1/2 \end{bmatrix} \qquad [\textbf{1 mark}]$$

which has eigenvalue $2, 1/2$ [**1 mark**].

(b) The unstable eigenvector is $\mathbf{v}_- = [1,0]^T$ [**1 mark**]. The stable eigenvector solves $\frac{3}{2}x + 3y = 0$ [**1 mark**], so we can choose $\mathbf{v}_+ = [-2,1]^T$ [**1 mark**].

(c) We have that $x = u^- - 2u^+$ and $y = u^+$ [**1 mark**]. Thus $u^+ = y$ and $u^- = x + 2y$ and $x - y = u^- - 3u^+$ [**1 mark**]. (DS) is transformed into

$$\begin{bmatrix} u^+_{n+1} \\ u^-_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}y_n + (x_n - y_n)^2 \\ 2x_n + 3y_n - (x_n - y_n)^2 + 2(\frac{1}{2}y_n + \frac{1}{2}(x_n - y_n)^2) \end{bmatrix} \qquad [\textbf{1 mark}]$$

$$= \begin{bmatrix} \frac{1}{2}u^+_n + (3u^+_n - u^-_n)^2 \\ 2u^-_n \end{bmatrix} \qquad [\textbf{1 mark}]$$

(d) By inspection $u^- = 0$ is invariant, hence $W^+_{loc} = \{(u^+, 0)\}$ [**1 mark**]. On the other hand, assume that $u^+ = g(u^-) = a_2(u^-)^2 + \cdots$ is the local unstable manifold expressed as the graph of a function up to second order [**1 mark**]. Then

$$\begin{aligned} u^+_{n+1} &= \frac{1}{2}u^+_n + (3u^+_n - u^-_n)^2 && \text{using part (c)} \\ &= \frac{1}{2}g(u^-_n) + (3g(u^-_n) - u^-_n)^2 && \text{using } u^+ = g(u^-) \\ &= (\frac{1}{2}a_2 + 1)(u^-_n)^2 + \cdots && \text{expanding } g(u^-)[\textbf{1 mark}] \end{aligned}$$

while,

$$\begin{aligned} u^+_{n+1} &= g(u^-_{n+1}) && \text{using invariance of } W^+_{loc} \\ &= g(2u^-_n) && \text{using part (c)} \\ &= 4a_2(u^-_n)^2 + \cdots && \text{using } g(u^-) = a_2(u^-)^2 + \cdots [\textbf{2 marks}] \end{aligned}$$

We equate coefficients and deduce

$$a_2 = \frac{2}{7} \qquad [\textbf{1 mark}].$$

Thus,

$$W^-_{loc} = \{(2(u^-)^2/7, u^-)\} \qquad [\textbf{2 marks}].$$

(e) Mark scheme for sketch:
(i) Correct labels [**2 marks**].
(ii) Correct orientation of manifolds [**1 mark**].
(iii) Correct arrows [**1 mark**].

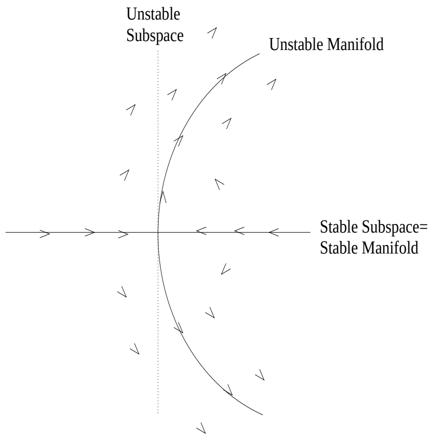

FIGURE 1. The stable and unstable manifolds of (DS).

(4) (a) State Sharkovskii's theorem. **/5**

(b) Let f : $[0,1] \to [0,1]$ be a continuous, surjective function whose graph is shown in figure 2. Prove: for each positive integer n, f has a periodic orbit of prime period n. **/8**

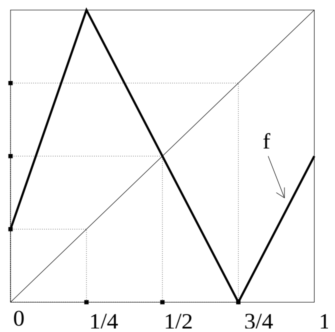

FIGURE 2. f : $[0,1] \to [0,1]$.

(c) Let $x_{n+1} = f_\mu(x_n)$ where $f_\mu(x) = x + \mu + x^2$ for $x, \mu \in \mathbb{R}$.

(i) Find the fixed points of this dynamical system. **/2**

(ii) Find the value of μ for which there is a saddle-node bifurcation. **/1**

(iii) Find the value of μ for which there is a flip bifurcation. Is it super- or sub-critical? **/6**

(iv) Sketch the bifurcation diagram in the (μ, x) plane. **/3**

Solution.

(a) Define the an ordering $\lhd$ on the positive integers by **[1 mark]**

$$3 \lhd 5 \lhd 7 \lhd \cdots \lhd 2\cdot3 \lhd 2\cdot5 \lhd 2\cdot7 \lhd \cdots \lhd 2^n\cdot3 \lhd 2^n 5 \lhd 2^n 7 \lhd \cdots\cdots \lhd 2^n \lhd \cdots \lhd 2^2 \lhd 2^1 \lhd 2^0,$$

[2 marks] where we enumerate all odd primes in increasing order, then twice the odd primes, and so on, and finally all powers of 2.
Recall that a point x has prime period n if it is a fixed point of f^n andnot of f^k for any $k < n$ **[1 mark]**.
Sharkovskii's Theorem. Let $f : [0,1] \to [0,1]$ be a continuous map. If f has a periodic point of prime period n, then f has a periodic point of prime period k for all $n \lhd k$ **[1 mark]**.

(b) By Sharkovskii's theorem, it suffices to prove that fhas a periodic point of prime period 3 **[1 mark]**.
Let $I_0 = [0,1/4]$, $I_1 = [1/4,3/4]$ and $I_2 = [3/4,1]$ **[1 mark]**. By inspection of the graph of fit is clear that fis continuous and
(i) f$(I_0) \supset I_1$;
(ii) f$(I_1) \supset I_2$;
(iii) f$(I_2) \supset I_0$ **[1 mark]**.
Therefore, by the IVT, there are intervals $K_i \subset I_i$ such that
(i) f$(K_0) = I_1$;
(ii) f$(K_1) = I_2$;
(iii) f$(K_2) = K_0$ **[1 mark]**.
Consequently, f^3 maps K_0 to itself **[1 mark]**. Therefore, f^3 has a fixed point in K_0, call it z **[1 mark]**.
If z does not have prime period 3 for f, then it must

be a fixed point of f [**1 mark**]. Then $z = \mathrm{f}(z)$ so z must lie in both K_0 and K_1, and $z = \mathrm{f}^2(z)$ so z must lie in K_1 and K_2, too. But $K_0 \subset I_0$ and $K_2 \subset I_2$ are disjoint [**1 mark**]. Absurd. Therefore, z has prime period 3. QED

(c) (i) Solving $f_\mu(x) = x$, we get $x = \pm\sqrt{-\mu}$ [**1 mark**]. Thus, we have a f.p. iff $\mu \leq 0$ [**1 mark**].

(ii) At $\mu = 0$, the f.p.s collide and disappear. This is a saddle-node bifurcation [**1 mark**].

(iii) There is a flip bifurcation at the parameter μ_* and f.p. x_* if $f'_{\mu_*}(x_*) = -1$ and $f'_\mu(x_\mu)$ passes through -1 at the same time [**2 marks**]. In this case $f'_\mu(x) = 1+2x$ which is positive for the positive f.p. and is -1 for the negative f.p. when $x_* = -1$ or $\mu_* = -1$. It is clear that the derivative moves through -1 as μ decreases. Therefore, there is a flip bifurcation only at $\mu_* = -1$ and $x_* = -1$ [**2 marks**].
To determine super-/sup-criticality, we use the Schwartzian derivative:

$$D_s\{f\} = \frac{f'''}{f'} - \frac{3}{2}\left[\frac{f''}{f'}\right]^2. \qquad [\mathbf{1\ mark}]$$

As $f'''_\mu = 0$, $f_\mu = 2$ and $f'_{\mu_*}(x_*) = -1$, we see that

$$D_s\{f_{\mu_*}\}(x_*) < 0.$$

Hence, the flip bifurcation is **supercritical** [**1 mark**].

(iv) Bifurcation diagram [**3 marks**]

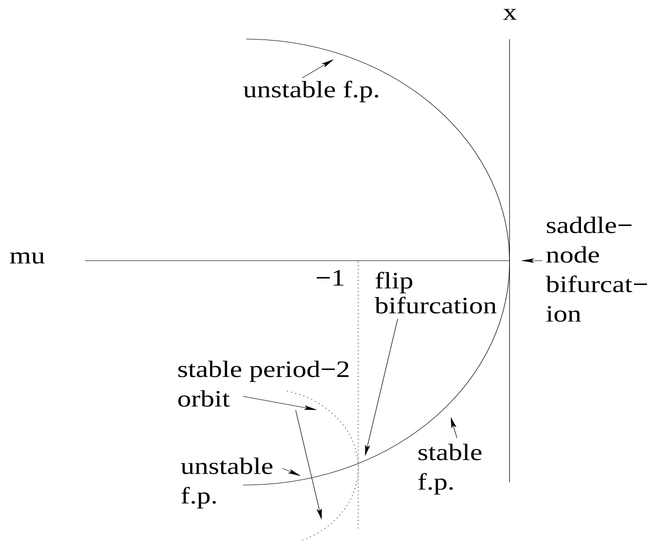

FIGURE 3. The bifurcation diagram for f_μ.