M4 Dynamical Systems

May 2006 — Questions

2 hours; best 3 answers count.

1. Consider the recurrence relation

$$x_{n+2} = \frac{7}{16}x_n + \frac{3}{2}x_{n+1} + x_n^2\,,$$

where $x_i \in \mathbb{R}$ $(i = 0, 1, 2, \ldots)$.

(a) Find the fixed points of this recurrence relation.

(b) Determine the nature of the fixed point at the (x, y) origin of the corresponding $\mathbb{R}^2$ system in which $y_n = x_{n+1}$.

(c) Give the definitions of the stable and unstable subspaces of the origin.

(d) Let $\begin{pmatrix} 1 \\ \alpha \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \beta \end{pmatrix}$ be vectors aligned with the stable and unstable subspaces, respectively. Find the constants α and β, and write down the equations of the stable and unstable subspaces.

(e) Introduce the vector $\begin{pmatrix} u_n^+ \\ u_n^- \end{pmatrix}$ which is defined via

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} u_n^+ \\ u_n^- \end{pmatrix}.$$

Thereby, show that the system may be expressed in the form

$$\begin{aligned} u_{n+1}^+ &= c_1\, u_n^+ + c_2\, \left(u_n^+\right)^2 + c_3\, \left(u_n^+ u_n^-\right) + c_4\, \left(u_n^-\right)^2\,, \\ u_{n+1}^- &= d_1\, u_n^- + d_2\, \left(u_n^+\right)^2 + d_3\, \left(u_n^+ u_n^-\right) + d_4\, \left(u_n^-\right)^2\,; \end{aligned}$$

and evaluate the constants $c_{1,\ldots,4}$ and $d_{1,\ldots,4}$.

(f) State the stable manifold theorem.

(g) Find the quadratic approximations to the stable and unstable manifolds in the (u^+, u^-) plane.

(h) Sketch the stable and unstable manifolds in the (u^+, u^-) plane. Include in your sketch a few representative orbits and identify the stable and unstable subspaces.

2. Consider the mapping $x_{n+1} = G(x_n)$ with

$$G(x) = 2\pi \sin x \,,$$

for $x \in [0, \pi]$.

(a) Let $I_1 = [0, a]$ and $I_2 = [b, \pi]$ with $a < b$. Find the largest value of a and smallest value of b such that G maps $I_1 \cup I_2$ onto $[0, \pi]$.

(b) Briefly discuss the stability of the fixed points of G.
(You are not required to locate exactly the fixed points).

(c) Describe the invariant set $S = \{x \in [0, \pi] : G^k(x) \in [0, \pi],\, k \in \mathbb{N}\}$ in terms of I_1 and I_2.

(d) Give the definition of the itinerary map which forms the basis of the symbolic dynamics on S for $G(x)$.

(e) Show that the itinerary map is

i. surjective (the *Nested Intervals Theorem* may be assumed);

ii. injective.

(f) How many prime period–6 orbits for G are in S?
How many prime period–8 orbits for G are in S?
Justify your answers.

(g) Show that the Liapunov exponent λ for G acting on S has the lower bound

$$\lambda \geq \ln \pi + \frac{\ln 3}{2}.$$

What can you infer from this?

3. (a) Prove that if a continuous $\mathbb{R}^1$ mapping has a period–3 orbit then it has prime period–n orbits for all $n \in \mathbb{N}$. (The *Intermediate Value Theorem* may be assumed).

   (b) Consider the $\mathbb{R}^1$ mapping $x_{n+1} = H_\mu(x_n)$ where

$$H_\mu(x) = \mu\left(\frac{1}{2} - \left|x - \frac{1}{2}\right|\right),$$

   and $\mu > 0$.

   i. For both $\mu > 1$ and $\mu < 1$, find the fixed points of $H_\mu(x)$ and discuss their stability.

   ii. Let $\mu = 2$.

      A. Find the orbit which starts at $x_0 = 2/7$.
      What can you infer from this?

      B. By considering the graph of the iterated map $H_2 \circ H_2$ or otherwise, find a period-2 orbit.

4. (a) Consider the $\mathbb{R}^1$ mapping $x_{n+1} = F_\nu(x_n)$ where

$$F_\nu(x) = \nu x \exp(-x)\,,$$

with $\nu \in \mathbb{R}$.

i. Find all the fixed points of F_ν.
ii. Determine the stability of the fixed points.
iii. Sketch the corresponding bifurcation diagram in the (ν, x) plane. Indicate the stability of the fixed points on your sketch.
iv. Does F_ν undergo subcritical flip bifurcations? Justify your answer.
v. Does the iterated mapping $F_\nu \circ F_\nu$ undergo subcritical flip bifurcations? Justify your answer. (Properties of the Schwarzian derivative presented in the lectures may be assumed).

(b) Consider the 2–dimensional continuous system governed by

$$\left.\begin{array}{l} \dot{r} = r(1 - r^2) \\ \dot{\theta} = 1 \end{array}\right\},$$

where $r(t) \in \Sigma$ and $\theta(t) > 0$ are the time–dependent plane polar coordinates, and Σ is the open unit interval $(0, 1)$. The trajectory which starts at

$$\left.\begin{array}{l} r(0) = r_0 \\ \theta(0) = 0 \end{array}\right\},$$

next crosses Σ at r_1.

i. Show that r_1 satisfies

$$\int_{r_0}^{r_1} \frac{1}{r(1 - r^2)}\, dr = 2\pi.$$

ii. Hence find the corresponding Poincaré map $P : \Sigma \to \Sigma$ such that $P(r_0) = r_1$.

[End of Paper]

Notes on Dynamical Systems questions

- Question 1: Parts (c) and (f) are bookwork. The other parts are standard — similar to tutorial questions.
- Question 2: We went through a similar example in class using the logistic map. The proofs required in part (e) are very similar to those given in class.
- Question 3: Part (a) is bookwork. Part (b) is a standard calculation — similar to tutorial questions.
- Question 4: Standard calculation — similar to tutorial questions.