M4 Dynamical Systems

May 2006 — Questions and Answers

M445/M06 **Dynamical Systems**

1. Consider the recurrence relation

$$x_{n+2} = \frac{7}{16}x_n + \frac{3}{2}x_{n+1} + x_n^2 \,,$$

where $x_i \in \mathbb{R}$ $(i = 0, 1, 2, \ldots)$.

(a) Find the fixed points of this recurrence relation.

(b) Determine the nature of the fixed point at the (x, y) origin of the corresponding $\mathbb{R}^2$ system in which $y_n = x_{n+1}$.

(c) Give the definitions of the stable and unstable subspaces of the origin.

(d) Let $\begin{pmatrix} 1 \\ \alpha \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \beta \end{pmatrix}$ be vectors aligned with the stable and unstable subspaces, respectively. Find the constants α and β, and write down the equations of the stable and unstable subspaces.

(e) Introduce the vector $\begin{pmatrix} u_n^+ \\ u_n^- \end{pmatrix}$ which is defined via

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} u_n^+ \\ u_n^- \end{pmatrix}.$$

Thereby, show that the system may be expressed in the form

$$\begin{aligned} u_{n+1}^+ &= c_1\, u_n^+ + c_2\, \left(u_n^+\right)^2 + c_3\, \left(u_n^+ u_n^-\right) + c_4\, \left(u_n^-\right)^2 \,, \\ u_{n+1}^- &= d_1\, u_n^- + d_2\, \left(u_n^+\right)^2 + d_3\, \left(u_n^+ u_n^-\right) + d_4\, \left(u_n^-\right)^2 \,; \end{aligned}$$

and evaluate the constants $c_{1,\ldots,4}$ and $d_{1,\ldots,4}$.

(f) State the stable manifold theorem.

(g) Find the quadratic approximations to the stable and unstable manifolds in the (u^+, u^-) plane.

(h) Sketch the stable and unstable manifolds in the (u^+, u^-) plane. Include in your sketch a few representative orbits and identify the stable and unstable subspaces.

Solution.

(a) The fixed points $x = x^*$ satisfy

$$x^* = \frac{7}{16}x^* + \frac{3}{2}x^* + (x^*)^2 \qquad \text{i.e.,} \quad x^*\left(x^* + \frac{15}{16}\right) = 0$$

so we have $x^* = 0$ and $x^* = -15/16$.

(b) We have

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mathbf{F}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} y_n \\ \frac{7}{16}x_n + \frac{3}{2}y_n + x_n^2 \end{pmatrix}.$$

The Jacobian matrix:

$$D\mathbf{F} = \begin{pmatrix} 0 & 1 \\ \frac{7}{16} + 2x_n & \frac{3}{2} \end{pmatrix} \qquad \Rightarrow \qquad D\mathbf{F}|_{\mathbf{0}} = \begin{pmatrix} 0 & 1 \\ \frac{7}{16} & \frac{3}{2} \end{pmatrix} = A.$$

The eigenvalues $\lambda = \lambda_{1,2}$ of A are given by

$$(0-\lambda)\left(\frac{3}{2} - \lambda\right) - \frac{7}{16} = 0, \qquad \text{i.e.,} \quad \left(\lambda - \frac{7}{4}\right)\left(\lambda + \frac{1}{4}\right) = 0;$$

thus, we have $\lambda_1 = -1/4$ and $\lambda_2 = 7/4$. Since $|\lambda_1| < 1$ and $|\lambda_2 > 1$, the fixed point at $x = 0$, $y = 0$ is a saddle point.

(c) The stable subspace E^+ is the span of eigenvectors associated with eigenvectors λ with $|\lambda| < 1$. The unstable subspace E^- is the span of eigenvectors associated with eigenvectors λ with $|\lambda| > 1$.

(d) The eigenvector $\begin{pmatrix} e_{x1} \\ e_{y1} \end{pmatrix}$ associated with λ_1 is given via

$$\begin{pmatrix} 0 & 1 \\ \frac{7}{16} & \frac{3}{2} \end{pmatrix}\begin{pmatrix} e_{x1} \\ e_{y1} \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} e_{x1} \\ e_{y1} \end{pmatrix}, \qquad \text{i.e.,} \quad \begin{pmatrix} e_{x1} \\ e_{y1} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{4} \end{pmatrix}, \text{ say;}$$

and the eigenvector $\begin{pmatrix} e_{x2} \\ e_{y2} \end{pmatrix}$ associated with λ_2 is given via

$$\begin{pmatrix} 0 & 1 \\ \frac{7}{16} & \frac{3}{2} \end{pmatrix}\begin{pmatrix} e_{x2} \\ e_{y2} \end{pmatrix} = \frac{7}{4}\begin{pmatrix} e_{x2} \\ e_{y2} \end{pmatrix}, \qquad \text{i.e.,} \quad \begin{pmatrix} e_{x2} \\ e_{y2} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{7}{4} \end{pmatrix}, \text{ say.}$$

Hence, E^+ is the line $y = -\frac{1}{4}x$, and E^- is the line $y = \frac{7}{4}x$.

(e) We have $S = \begin{pmatrix} 1 & 1 \\ -\frac{1}{4} & \frac{7}{4} \end{pmatrix}$ with

$$\begin{pmatrix} u_n^+ \\ u_n^- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{7}{4} & -1 \\ \frac{1}{4} & 1 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{7}{4}x_n - y_n \\ \frac{1}{4}x_n + y_n \end{pmatrix} \tag{1}$$

and

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\frac{1}{4} & \frac{7}{4} \end{pmatrix} \begin{pmatrix} u_n^+ \\ u_n^- \end{pmatrix} = \begin{pmatrix} u_n^+ + u_n^- \\ -\frac{1}{4}u_n^+ + \frac{7}{4}u_n^- \end{pmatrix}. \tag{2}$$

Now, using eqn (1), we can re-write $x_{n+1} = y_n$ as

$$u_{n+1}^+ + u_{n+1}^- = -\frac{1}{4}u_n^+ + \frac{7}{4}u_n^-; \tag{3}$$

and, using eqn (2), we can re-write $y_{n+1} = \frac{7}{16}x_n + \frac{3}{2}y_n + x_n^2$ as

$$\begin{aligned} -\frac{1}{4}u_{n+1}^+ + \frac{7}{4}u_{n+1}^- &= \frac{7}{16}\left(u_n^+ + u_n^-\right) + \frac{3}{2}\left(-\frac{1}{4}u_n^+ + \frac{7}{4}u_n^-\right) + \left(u_n^+ + u_n^-\right)^2 \\ &= \frac{1}{16}u_n^+ + \frac{49}{16}u_n^- + \left(u_n^+ + u_n^-\right)^2. \end{aligned} \tag{4}$$

Taking $\frac{7}{4}$(3) - (4) gives:

$$\begin{aligned} 2u_{n+1}^+ &= -\frac{7}{16}u_n^+ + \frac{49}{16}u_n^- - \frac{1}{16}u_n^+ - \frac{49}{16}u_n^- - \left(u_n^+ + u_n^-\right)^2 \\ \text{i.e.,} \quad u_{n+1}^+ &= -\frac{1}{4}u_n^+ - \frac{1}{2}\left(u_n^+ + u_n^-\right)^2; \end{aligned} \tag{5}$$

i.e., we have $c_1 = -\frac{1}{4}$, $c_2 = -\frac{1}{2}$, $c_3 = -1$ and $c_4 = -\frac{1}{2}$.
Similarly, taking $\frac{1}{4}$(3) + (4) gives:

$$\begin{aligned} 2u_{n+1}^- &= -\frac{1}{16}u_n^+ + \frac{7}{16}u_n^- + \frac{1}{16}u_n^+ + \frac{49}{16}u_n^- + \left(u_n^+ + u_n^-\right)^2 \\ \text{i.e.,} \quad u_{n+1}^- &= \frac{7}{4}u_n^- + \frac{1}{2}\left(u_n^+ + u_n^-\right)^2; \end{aligned} \tag{6}$$

i.e., we have $d_1 = \frac{7}{4}$, $d_2 = \frac{1}{2}$, $d_3 = 1$ and $d_4 = \frac{1}{2}$.

(f) The stable manifold theorem asserts the existence of (i) an invariant stable manifold W^s defined by $\left\{\mathbf{x} : \mathbf{F}^k(\mathbf{x}) \to \mathbf{0} \quad \text{as} \quad k \to \infty\right\}$; and (ii) an invariant unstable manifold W^u defined by $\left\{\mathbf{x} : \mathbf{F}^k(\mathbf{x}) \to \mathbf{0} \quad \text{as} \quad k \to -\infty\right\}$.

(g) Consider the quadratic expansion of the stable manifold: $u^- = a_2\left(u^+\right)^2$. We have

$$\begin{aligned} u_{n+1}^- &= a_2\left(u_{n+1}^+\right)^2 \\ &= a_2\left[-\frac{1}{4}u_n^+ - \frac{1}{2}\left(u_n^+ + u_n^-\right)^2\right]^2 && \text{from eqn (5)} \\ &= a_2\left[\frac{1}{16}\left(u_n^+\right)^2 + \frac{1}{4}u_n^+\left(u_n^+ + u_n^-\right)^2 + \frac{1}{4}\left(u_n^+ + u_n^-\right)^4\right] \\ &\approx a_2\frac{1}{16}\left(u_n^+\right)^2 && \text{(neglecting cubic and higher-order terms).} \end{aligned} \tag{7}$$

Also, recalling eqn (6), we have

$$\begin{aligned} u_{n+1}^- &= \frac{7}{4}a_2\left(u_n^+\right)^2 + \frac{1}{2}\left[u_n^+ + a_2\left(u_n^+\right)^2\right]^2 \\ &\approx \left(\frac{7}{4}a_2 + \frac{1}{2}\right)\left(u_n^+\right)^2 \qquad \text{(neglecting cubic and higher-order terms).} \end{aligned} \tag{8}$$

Comparing eqns (7) and (8), we find

$$a_2\frac{1}{16} = \frac{7}{4}a_2 + \frac{1}{2}, \qquad \text{i.e.,} \quad a_2 = -\frac{8}{27}.$$

Consider the quadratic expansion of the unstable manifold: $u^+ = b_2\left(u^-\right)^2$. We have

$$\begin{aligned} u_{n+1}^+ &= b_2\left(u_{n+1}^-\right)^2 \\ &= b_2\left[\frac{7}{4}u_n^- + \frac{1}{2}\left(u_n^+ + u_n^-\right)^2\right]^2 \qquad \text{from eqn (6)} \\ &= b_2\left[\frac{49}{16}\left(u_n^-\right)^2 + \frac{7}{4}u_n^-\left(u_n^+ + u_n^-\right)^2 + \frac{1}{4}\left(u_n^+ + u_n^-\right)^4\right] \\ &\approx b_2\frac{49}{16}\left(u_n^-\right)^2 \qquad \text{(neglecting cubic and higher-order terms).} \end{aligned} \tag{9}$$

Also, recalling eqn (5), we have

$$\begin{aligned} u_{n+1}^+ &= -\frac{1}{4}b_2\left(u_n^-\right)^2 - \frac{1}{2}\left[u_n^- + b_2\left(u_n^-\right)^2\right]^2 \\ &\approx \left(-\frac{1}{4}b_2 - \frac{1}{2}\right)\left(u_n^-\right)^2 \qquad \text{(neglecting cubic and higher-order terms).} \end{aligned} \tag{10}$$

Comparing eqns (9) and (10), we find

$$b_2\frac{49}{16} = -\frac{1}{4}b_2 - \frac{1}{2}, \qquad \text{i.e.,} \quad b_2 = -\frac{8}{53}.$$

(h) Sketch see figure 1:

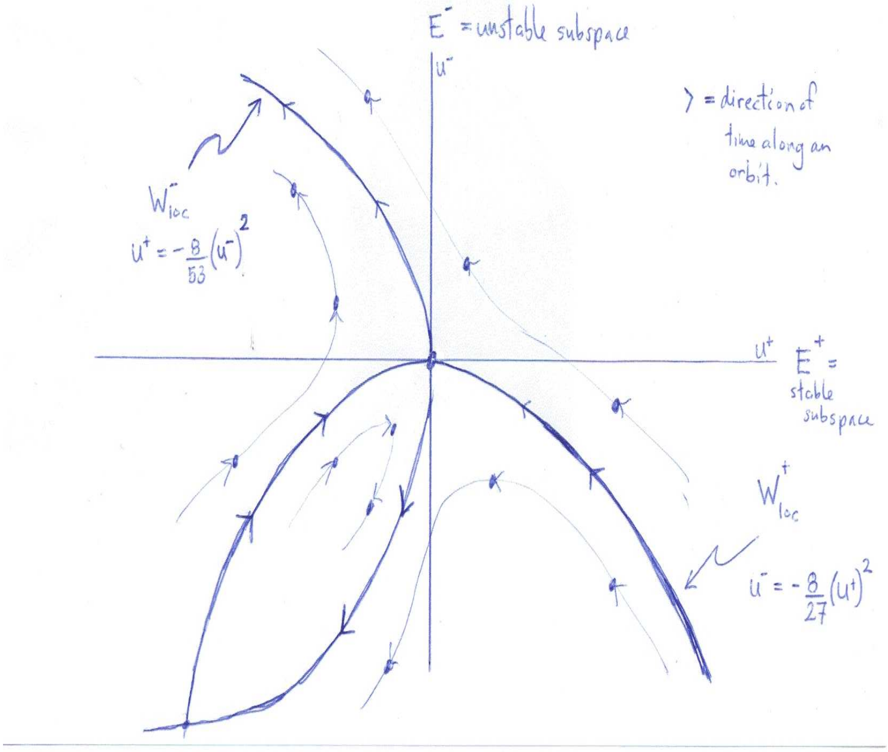

Figure 1: Stable and unstable manifolds.

2. Consider the mapping $x_{n+1} = G(x_n)$ with

$$G(x) = 2\pi \sin x \,,$$

for $x \in [0, \pi]$.

   (a) Let $I_1 = [0, a]$ and $I_2 = [b, \pi]$ with $a < b$. Find the largest value of a and smallest value of b such that G maps $I_1 \cup I_2$ onto $[0, \pi]$.

   (b) Briefly discuss the stability of the fixed points of G.
   (You are not required to locate exactly the fixed points).

   (c) Describe the invariant set $S = \{x \in [0, \pi] : G^k(x) \in [0, \pi],\, k \in \mathbb{N}\}$ in terms of I_1 and I_2.

   (d) Give the definition of the itinerary map which forms the basis of the symbolic dynamics on S for $G(x)$.

   (e) Show that the itinerary map is

      i. surjective (the *Nested Intervals Theorem* may be assumed);

      ii. injective.

   (f) How many prime period–6 orbits for G are in S?
   How many prime period–8 orbits for G are in S?
   Justify your answers.

(g) Show that the Liapunov exponent λ for G acting on S has the lower bound

$$\lambda \geq \ln \pi + \frac{\ln 3}{2}.$$

What can you infer from this?

Solution.

(a) $\sin \frac{\pi}{6} = \sin \frac{5\pi}{6} = \frac{1}{2} \qquad \Rightarrow \qquad a = \frac{\pi}{6}$ and $b = \frac{5\pi}{6}$

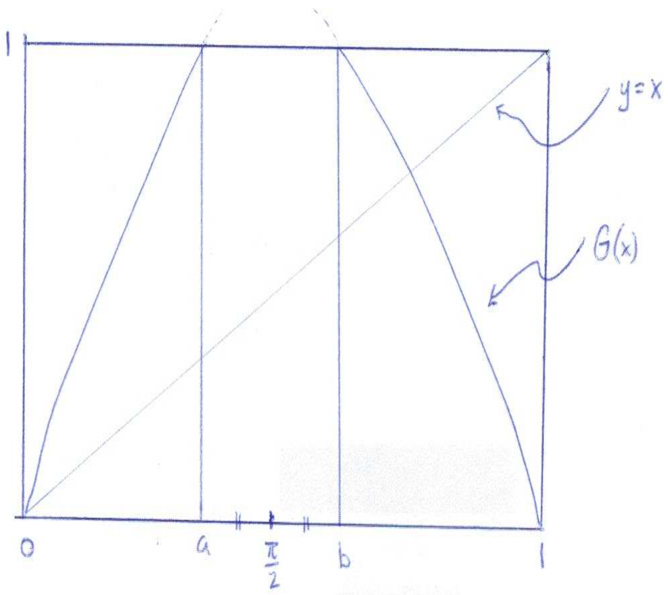

Figure 2: Graph of $G(x) = 2\pi \sin(x)$.

(b) From the sketch (see figure 2) there are two fixed points: $x_i = 0$ and $x_{ii} \in (\frac{5\pi}{6}, \pi)$.

$$\begin{aligned}
G'(0) = 2\pi \cos 0 = 2\pi &\Rightarrow |G'(x_i)| > 1 \\
&\Rightarrow x_i \text{ is unstable} \\
G'(\frac{5\pi}{6}) = 2\pi \cos \frac{5\pi}{6} = -\sqrt{3}\pi &\Rightarrow |G'(\frac{5\pi}{6})| > 1 \\
G'(\pi) = 2\pi \cos \pi = -2 &\Rightarrow |G'(\pi)| > 1 \\
\text{and } G \text{ is strictly decreasing on } (\tfrac{5\pi}{6}, \pi) &\Rightarrow x_{ii} \text{ is unstable}
\end{aligned}$$

(c) Construct the sets:

$$I_{j_0 j_1 j_2 \cdots j_n} = \left\{ x : \quad x \in I_{j_0} \quad \text{and} \quad G^k(x) \in I_{j_k} \quad \text{for} \quad 1 \leq k \leq n \right\}$$

where $j_k = 1$ or 2; and

$$S_n = \bigcup_{j_0, j_1, \cdots, j_n} I_{j_0 j_1 j_2 \cdots j_n}.$$

Then, the invariant set is given by $S = \cap_{n=0}^{\infty} S_n$.

(d) The symbolic dynamics are constructed via the itinerary map $h : S \to \Sigma$ where $\Sigma = \{1, 2\}^{\mathbb{N}}$ (i.e., Σ is the set of all sequences of the symbols '1' and '2') and

$$h(x) = (j_0, j_1, j_2, \ldots) \quad \text{with} \quad x_0 \in I_{j_0} \quad \text{and} \quad G^k(x) \in I_{j_k}.$$

(e) i. Consider the sequence of symbols $(j_0, j_1, j_2, \cdots) \in \Sigma$. Form the sequence of sets $J_n = I_{j_0 j_1 j_2 \cdots j_n}$. Since the J_n form a nested family of closed intervals, by the Nested Intervals Theorem, $\exists x^* \in \cap_{n=0}^{\infty} J_n$. Thus, $h(x^*) = (j_0, j_1, j_2, \cdots)$; i.e., h is surjective.

ii. Observe that for $x \in S$ we have

$$|G'(x)| = |2\pi \cos x| \geq |2\pi \cos \frac{\pi}{6}| > 1.$$

Therefore, $|\left(G^k\right)'(x)| > 1$ in S, since $\left(G^n\right)'(x_0) = \prod_{i=0}^{n-1} G'(x_i)$.

Suppose there exists $x, y \in S$ and $h(x) = h(y)$. This implies that $G^k(x)$ and $G^k(y)$ are in the same subinterval $I_{j_0 j_1 j_2 \cdots j_k}$ of the pair $I_1 \cup I_2$. From $|G'| \geq q > 1$, we have that

$$\begin{aligned} \left|\frac{G^{k+1}(y) - G^{k+1}(x)}{G^k(y) - G^k(x)}\right| &\geq q \\ \text{i.e.,} \qquad \left|G^{k+1}(y) - G^{k+1}(x)\right| &\geq q\left|G^k(y) - G^k(x)\right| \\ &\geq q^2\left|G^{k-1}(y) - G^{k-1}(x)\right| \\ &\geq q^3\left|G^{k-2}(y) - G^{k-2}(x)\right| \\ &\vdots \\ &\geq q^{k+1}|y - x|. \end{aligned}$$

But $|G^{k+1}(y) - G^{k+1}(x)| \leq 1$ (since $x, y \in S$). So

$$|y - x| \leq \frac{1}{q^{k+1}}$$

which implies that $x = y$. Hence, h is injective.

(f) Since h is bijective, a k–periodic orbit of G corresponds to a fixed point of σ^k

where σ is the backward shift map.

$$\begin{aligned}\text{no. of prime period-6 orbits of } G &= \frac{1}{6}\Big(\text{no. of fixed points of } \sigma^6 \text{ - no. of fixed points of } \sigma^3 \\ &\quad -\text{no. of fixed points of } \sigma^2 + \text{no. of fixed points of } \sigma\Big) \\ &= \frac{2^6 - 2^3 - 2^2 + 2}{6} = 9.\end{aligned}$$

$$\begin{aligned}\text{no. of prime period-8 orbits of } G &= \frac{1}{8}\Big(\text{no. of fixed points of } \sigma^8 \text{ - no. of fixed points of } \sigma^4\Big) \\ &= \frac{2^8 - 2^4}{8} = 30.\end{aligned}$$

(g) For $x \in S$ we have

$$|G'(x)| \geq |G'(\frac{\pi}{6})| = \pi\sqrt{3}.$$

Therefore, Liapunov exponent

$$\lambda = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{i=0}^{n-1}\ln|G'(x_i)|\right\} \geq \ln(\pi\sqrt{3})\lim_{n\to\infty}\left\{\frac{1}{n}\sum_{i=0}^{n-1}1\right\} = \ln(\pi\sqrt{3}) = \ln\pi + \frac{1}{2}\ln 3\,;$$

Since $\lambda > 0$, there is a sensitive dependency upon initial conditions.

3. (a) Prove that if a continuous $\mathbb{R}^1$ mapping has a period–3 orbit then it has prime period–n orbits for all $n \in \mathbb{N}$. (The *Intermediate Value Theorem* may be assumed).

(b) Consider the $\mathbb{R}^1$ mapping $x_{n+1} = H_\mu(x_n)$ where

$$H_\mu(x) = \mu\left(\frac{1}{2} - \left|x - \frac{1}{2}\right|\right),$$

and $\mu > 0$.

i. For both $\mu > 1$ and $\mu < 1$, find the fixed points of $H_\mu(x)$ and discuss their stability.

ii. Let $\mu = 2$.

A. Find the orbit which starts at $x_0 = 2/7$.
What can you infer from this?

B. By considering the graph of the iterated map $H_2 \circ H_2$ or otherwise, find a period-2 orbit.

Solution.

(a) Suppose the mapping H has the period–3 orbit $\{a, b, c\}$. WLOG we assume $a < b < c$. To prove that H has period–n orbits for all $n \in \mathbb{N}$:

Let us define $I_0 = [a, b]$ and $I_1 = [b, c]$ and make the following observations

i. $H(I_0) \supseteq I_1$.

ii. $H(I_1) \supseteq I_0 \cup I_1$.

iii. If I is a closed interval and $H(I) \supseteq I$, then H has a fixed point in I.

iv. Suppose I, J are closed intervals. If $H(I) \supseteq J$, then there exists a closed interval $K \subseteq I$ such that $H(K) = J$.

The last two observations can be established using the intermediate value theorem.

- We start by noting that (ii) and (iii) imply that H has a fixed point in I_1. Also, (i), (ii) and (iii) imply that H^2 has a fixed point in I_0, so that H has a period-2 orbit. Thus, the period-1 and period-2 cases are proven.
- We have $n > 3$. Now we construct a nested sequence of closed intervals A_n: let $A_0 = I_1$, (ii) and (iv) imply that there is a $A_1 \subseteq A_0$ with $H(A_1) = A_0 = I_1$. Similarly, there is a $A_2 \subseteq A_1$ with $H(A_2) = A_1$ and so $H^2(A_2) = A_0$. Proceeding similarly, the sequence

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_{n-2}, \quad \text{with} \quad H^k(A_k) = A_0, \quad k = 1, 2, \ldots, n-2,$$

can be constructed. The next interval in the sequence, A_{n-1} is constructed by noting that $H^{n-1}(A_{n-2}) = H(A_0) \supseteq I_0$ (using (ii)). Then, (iv) implies that there is a $A_{n-1} \subseteq A_{n-2}$ with $H^{n-1}(A_{n-1}) = I_0$. Finally since $H^n(A_{n-1}) = H(I_0) \supseteq I_1$ (using (i)), there exists a $A_n \subseteq A_{n-1}$ with $H^n(A_n) = A_0 = I_1$. Now, by construction $A_n \subseteq A_0$, so that $H^n(A_n) \supseteq A_n$. So (iii) then implies that there exists a fixed point $x^\star \in A_n$ with $H^n(x^\star) = x^\star$. This is a prime period-n point unless it is also fixed point of H^k for $k < n$. But this is impossible since $x^\star \in A_k,\ k = 0, 1, \cdots, n$ gives that $H^k(x^\star) \in I_1$ for $k = 1, 2, \ldots, n-2$ and we also have $H^{n-1}(x^\star) \in I_0$. (The case $H^{n-1}(x^\star) \in I_0 \cap I_1 = \{b\}$ can be excluded since it would imply $n = 3$.)

(b) i.

$$H_\mu(x) = \begin{cases} \mu x, & \text{for } \ x \leq \frac{1}{2} \\ \mu\,(1 - x)\,, & \text{for } \ x > \frac{1}{2} \end{cases}$$

Sketch see figure 3.

- For $x \leq \frac{1}{2}$:
  Fixed points satisfy $x^* = \mu x^*$, so for $\mu < 1$ and $\mu > 1$ we have the fixed point $x^* = 0$. $G'_\mu(x) = \mu$ so x^* is stable for $\mu < 1$ and unstable for $\mu > 1$.

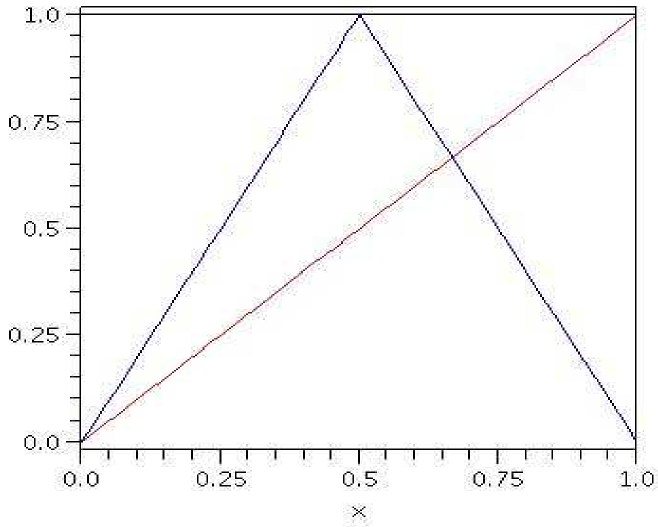

Figure 3: Graph of H_μ with $\mu = 2$.

- For $x > \frac{1}{2}$:
  Fixed points satisfy $x^* = \mu(1 - x^*)$; i.e., $x^* = \frac{\mu}{1+\mu}$. But $x^* > \frac{1}{2}$ so we require that $\frac{\mu}{1+\mu} > \frac{1}{2}$; i.e., $\mu > 1$. Since $G'_\mu(x) = -\mu$ the fixed point $x^* = \frac{\mu}{1+\mu}$ is unstable.

ii. A. Let $\mu = 2$. Consider $x_0 = \frac{2}{7}$. Then

$$
\begin{aligned}
& x_1 = H_\mu(x_0) = 2\left(\frac{1}{2} - |\frac{2}{7} - \frac{1}{2}|\right) = \frac{4}{7} \\
\text{and} \quad & x_2 = H_\mu(x_1) = 2\left(\frac{1}{2} - |\frac{4}{7} - \frac{1}{2}|\right) = \frac{6}{7} \\
\text{and} \quad & x_3 = H_\mu(x_2) = 2\left(\frac{1}{2} - |\frac{6}{7} - \frac{1}{2}|\right) = \frac{2}{7} \\
\text{i.e.,} \quad & x_0 = x_3\,.
\end{aligned}
$$

Therefore there is a period–3 orbit $\left\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right\}$.
By (a), H_2 has period–n orbits for all $n \in \mathbb{N}$.

B. Consider the iterated map $(H_2 \circ H_2)(x)$ (sketch see figure 4):

- For $x \le \frac{1}{2}$:

$$
H_2(H_2(x)) = \begin{cases} 2 \times 2x, & \text{provided} \quad 2x \le \frac{1}{2} \\ 2(1 - 2x), & \text{provided} \quad 2x > \frac{1}{2} \end{cases}
$$

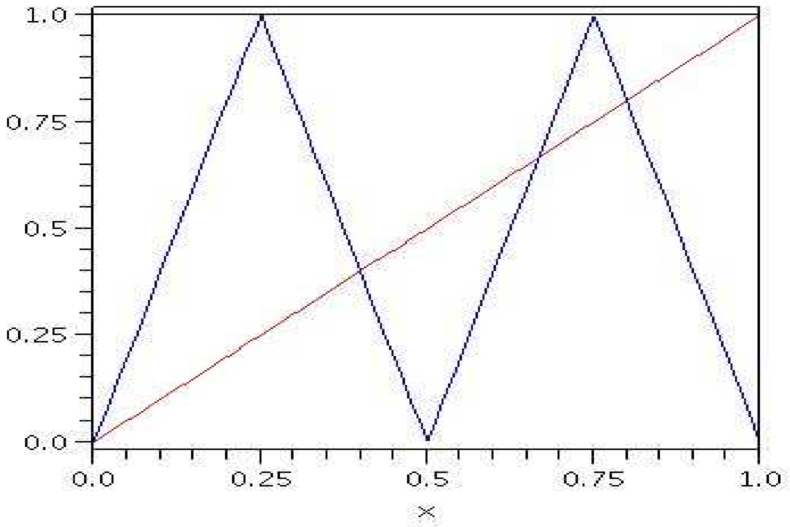

Figure 4:

- For $x > \frac{1}{2}$:

$$H_2\left(H_2(x)\right) = \begin{cases} 2 \times 2\,(1-x)\,, & \text{provided} \quad 2\,(1-x) \le \frac{1}{2} \\ 2\,(1-2(1-x))\,, & \text{provided} \quad 2\,(1-x) > \frac{1}{2} \end{cases}$$

A fixed point for $H_2 \circ H_2$ for $x \in \left(\frac{1}{4}, \frac{2}{2}\right)$ is found by solving $x^* = 2\,(1-2x^*)$; i.e., $x^* = \frac{2}{5}$. Now $H_2(\frac{2}{5}) = 2 \times \frac{2}{5} = \frac{4}{5}$ and $H_2(\frac{4}{5}) = 2\left(1 - \frac{4}{5}\right) = \frac{2}{5}$; i.e., $\left\{\frac{2}{5}, \frac{4}{5}\right\}$ is a period–2 orbit.

4. (a) Consider the $\mathbb{R}^1$ mapping $x_{n+1} = F_\nu(x_n)$ where

$$F_\nu(x) = \nu x \exp(-x)\,,$$

with $\nu \in \mathbb{R}$.

i. Find all the fixed points of F_ν.
ii. Determine the stability of the fixed points.
iii. Sketch the corresponding bifurcation diagram in the (ν, x) plane. Indicate the stability of the fixed points on your sketch.
iv. Does F_ν undergo subcritical flip bifurcations?
Justify your answer.
v. Does the iterated mapping $F_\nu \circ F_\nu$ undergo subcritical flip bifurcations?
Justify your answer. (Properties of the Schwarzian derivative presented in the lectures may be assumed).

(b) Consider the 2–dimensional continuous system governed by

$$\left.\begin{array}{l}\dot{r} = r(1-r^2)\\ \dot{\theta} = 1\end{array}\right\},$$

where $r(t) \in \Sigma$ and $\theta(t) > 0$ are the time–dependent plane polar coordinates, and Σ is the open unit interval $(0, 1)$. The trajectory which starts at

$$\left.\begin{array}{l}r(0) = r_0\\ \theta(0) = 0\end{array}\right\},$$

next crosses Σ at r_1.

i. Show that r_1 satisfies

$$\int_{r_0}^{r_1} \frac{1}{r(1-r^2)}\,dr = 2\pi.$$

ii. Hence find the corresponding Poincaré map $P : \Sigma \to \Sigma$ such that $P(r_0) = r_1$.

Solution.

(a) i. Fixed points satisfy $x^* = \nu x^* \exp(-x^*)$. Clearly $x^* = 0$ is a fixed point. For $x^* \neq 0$ we have $1 = \nu \exp(-x^*)$; i.e., $x^* = \ln\nu$ for $\nu > 0$. Therefore there are two fixed points:

$$\left.\begin{array}{lll}x_1^* = 0 & \text{exists} & \forall\nu\\ x_2^* = \ln\nu & \text{exists for} & \nu > 0\end{array}\right\}.$$

ii. Stability of fixed points is determined by

$$F_\nu'(x) = \nu\exp(-x) - \nu x \exp(-x)\,.$$

Then, $F_\nu'(x_1^*) = \nu$ and $F_\nu'(x_2^*) = \nu\exp(\ln\frac{1}{\nu}) - \nu\ln\nu\,\exp\left(\ln\frac{1}{\nu}\right) = 1 - \ln\nu$.

Hence, x_1^* is stable for $|\nu| < 1$ and unstable for $|\nu| > 1$. And x_2^* is stable provided that

$$\begin{array}{rl} & -1 < 1 - \ln\nu < 1\\ \Rightarrow & 0 < \ln\nu < 2\\ \text{i.e.,} & 1 < \nu < \exp(2)\end{array}$$

and unstable for $\nu < 1$ and $\nu > \exp(2)$

iii. Sketch: see figure 5.

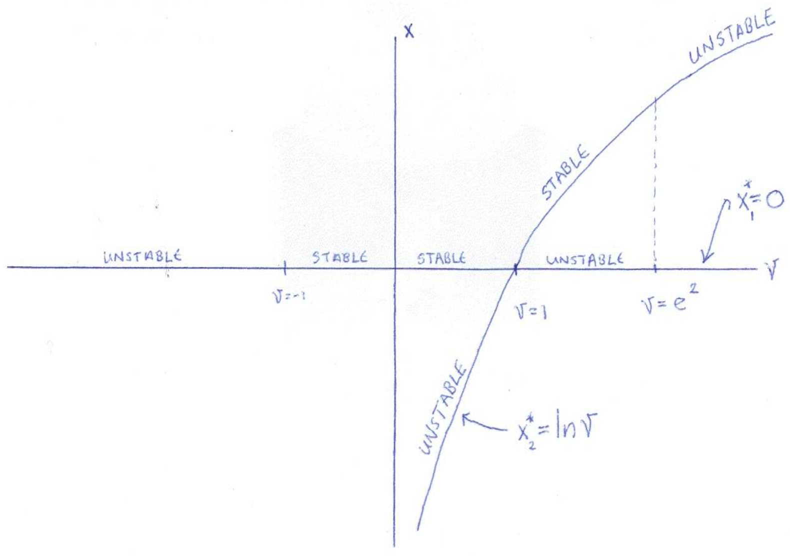

Figure 5:

iv. $F_\nu''(x) = -2\nu e^{-x} + \nu x e^{-x}$ and $F_\nu'''(x) = 3\nu e^{-x} - \nu x e^{-x}$
Schwarzian derivative:

$$\begin{aligned} D_s\{F_\nu\} &= \frac{F_\nu'''}{F_\nu'} - \frac{3}{2}\left(\frac{F_\nu''}{F_\nu'}\right)^2 \\ &= \frac{\nu e^{-x}(3-x)}{\nu e^{-x}(1-x)} - \frac{3}{2}\left[\frac{\nu e^{-x}(x-2)}{\nu e^{-x}(1-x)}\right]^2 \\ &= \frac{3-x}{1-x} - \frac{3}{2}\left(\frac{2-x}{1-x}\right)^2 \\ &= \frac{-(x-2)^2 - 2}{2(1-x)^2} \\ &< 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad x = \ln\nu \end{aligned}$$

Therefore, there are no subcritical flip bifurcations.

Notice that for flip bifurcation at fixed point $x = \ln\nu$, we have:
$F_\nu'(\ln\nu) = 1 - \ln\nu = -1 \Rightarrow \nu = e^2 \Rightarrow x = 2$.
Therefore, the Schwarzian derivative is well-defined at $x = \ln\nu$ (and obviously at $x = 0$ too).

v. Since

$$D_s\{(F_\mu \circ G_\mu)\}(x) = \left[G_\mu'(x)\right]^2 D_s\{F_\mu\}\left(G_\mu(x)\right) + D_s\{F_\mu\}(x),$$

for any three times differentiable F_ν and G_ν (a standard result proved in class), it follows that $D_s\{(F_\mu \circ F_\mu)\}(x) < 0$ and so there are no subcritical flip bifurcations.

(b) i. Suppose the trajectory starting at $(r_0, 0)$ next crosses Σ axis at time $t = T$; i.e.,

$$\left.\begin{array}{ll} r(0) = r_0, & r(T) = r_1 \\ \theta(0) = 0, & \theta(T) = 2\pi \end{array}\right\}.$$

Then

$$\dot{\theta} = 1 \qquad \Rightarrow \qquad \int_0^{2\pi} d\theta = \int_0^T dt\,, \qquad \text{i.e.,} \qquad 2\pi = T$$

and

$$\dot{r} = r\left(1 - r^2\right) \qquad \Rightarrow \qquad \int_{r_0}^{r_1} \frac{dr}{r\,(1-r^2)} = \int_0^T dt = 2\pi\,.$$

ii.

$$\begin{aligned} \int_{r_0}^{r_1} \frac{dr}{r\,(1-r^2)} &= \int_{r_0}^{r_1} \left(\frac{1}{r} - \frac{1}{2\,(1+r)} + \frac{1}{2\,(1-r)}\right) dr \\ &= \left[\ln r - \frac{1}{2}\ln(1+r) - \frac{1}{2}\ln(1-r)\right]_{r_0}^{r_1} \\ &= \left[\ln\left(\frac{r}{\sqrt{1-r^2}}\right)\right]_{r_0}^{r_1} \\ \Rightarrow \quad e^{2\pi} &= \frac{r_1}{\sqrt{1-r_1^2}}\,\frac{\sqrt{1-r_0^2}}{r_0} \\ \text{i.e.,} \quad \frac{r_1^2}{1-r_1^2} &= \frac{r_0^2 e^{4\pi}}{1-r_0^2} \\ \text{i.e.,} \quad \frac{1}{\frac{1}{r_1^2}-1} &= \frac{e^{4\pi}}{\frac{1}{r_0^2}-1} \\ \Rightarrow \quad r_1 &= \left[1 + e^{-4\pi}\left(r_0^{-2} - 1\right)\right]^{-1/2} \\ \text{i.e.,} \quad P(r) &= \left[1 + e^{-4\pi}\left(r^{-2} - 1\right)\right]^{-1/2}. \end{aligned}$$

[End of Paper]