M4 Dynamical Systems

May 2005 — Questions

2 hours; best 3 answers count.

1. Consider the two-dimensional system

$$x_{n+1} = -x_n + 3y_n - \frac{15}{8}\left(x_n - y_n\right)^3 ,$$
$$y_{n+1} = -\frac{3}{2}x_n + \frac{7}{2}y_n - \frac{15}{8}\left(x_n - y_n\right)^3 ,$$

where $x_i, y_i \in \mathbb{R}$.

(a) Show that there is a saddle-point at the origin.

(b) Give the definitions of the stable and unstable subspaces of the origin and find their equations.

(c) Introduce the vector $\begin{pmatrix} u_n^+ \\ u_n^- \end{pmatrix}$ which is defined via

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} \begin{pmatrix} u_n^+ \\ u_n^- \end{pmatrix},$$

where $\begin{pmatrix} a \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ b \end{pmatrix}$ are vectors aligned with the stable and unstable subspaces, respectively. Thereby, show that the system may be expressed in the form

$$u_{n+1}^+ = \alpha\, u_n^+ ,$$
$$u_{n+1}^- = \beta\, u_n^- + \gamma\, \left(u_n^+\right)^3 ;$$

and evaluate the constants a, b, α, β and γ.

(d) State the stable manifold theorem and show that

i. the stable manifold is given *exactly* by

$$u^- = \delta\, \left(u^+\right)^3 ;$$

ii. the unstable manifold is given *exactly* by

$$u^+ = \rho\, ;$$

and evaluate the constants δ and ρ.

(e) Sketch the stable and unstable manifolds in the (u^+, u^-) plane. Include in your sketch a few representative orbits and indentify the stable and unstable subspaces.

2. Consider the map

$$x_{n+2} = x_{n+1} - x_n + 2(2x_n - x_{n+1})^3\,, \qquad (*)$$

where $x_i \in \mathbb{R}$.

(a) Find the fixed points of this map.

(b) Use the corresponding linearized map to discuss the stability of the fixed points.

(c) Let $z_n = x_n + \varepsilon x_{n+1}$ where $\varepsilon \in \mathbb{C}$. Show that, by choosing ε appropriately, the system $(*)$ may expressed as

$$z_{n+1} = \alpha\, z_n + \beta_1\, (z_n)^3 + \beta_2\, (z_n)^2\, \overline{z}_n + \beta_3\, z_n\, (\overline{z}_n)^2 + \beta_4\, (\overline{z}_n)^3$$

and calculate the complex-valued constants α, β_1, β_2, β_3 and β_4.

(d) Explain briefly (without performing any calculations) how the system may rewritten in terms of a new variable ζ_n as

$$\zeta_{n+1} = \alpha\zeta_n + b\,\zeta_n^2\overline{\zeta}_n + O(|\zeta_n|^4)$$

and express b in terms of ε.

(e) Using (d), show how the stability of the origin depends on b/ε and thereby comment upon the stability of the origin for the system $(*)$.

3. Consider the map

$$H_\mu(x) = \mu \tan^{-1} x\,,$$

where x is a real-valued variable and μ is a real-valued parameter.

(a) How many fixed points are there? Specify the ranges of values of μ for which they exist.

(b) Calculate the Schwarzian derivative of H_μ.

(c) Describe the bifurcations which occur for

i. $\mu = 1$,

ii. $\mu = -1$.

If there are flip bifurcations, state whether they are supercritical or subcritical. You may wish to make use of the following Taylor series expansion:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\,,$$

for $|x| < 1$.

(d) Sketch the bifurcation diagram in the (μ, x) plane; indicate the stability of the fixed points in your diagram.

4. (a) Suppose a continuous mapping G has a period-3 orbit (a, b, c) where $a < b < c$.
   i. Show that G has orbits of period 1 and period 2.
   ii. Show that G has orbits of *prime* period n for all $n > 3$.

   (b) Consider the $\mathbb{R}^2$ map represented as

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n) \qquad (\dagger)$$

where

$$\mathbf{x}_{n+1} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}, \qquad \mathbf{x}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \qquad \mathbf{F}(\mathbf{x}_n) = \begin{pmatrix} y_n + \nu \\ y_n^2 - x_n^2 \end{pmatrix},$$

and ν is a real-valued parameter.

   i. Find the $\begin{pmatrix} x \\ y \end{pmatrix}$ fixed points of the map (†) in terms of ν.
   ii. If the map (†) undergoes a Hopf bifurcation, what can you infer about the eigenvalues of the Jacobian matrix of derivatives of $\mathbf{F}$?
   iii. Find the value of ν, and the corresponding $\begin{pmatrix} x \\ y \end{pmatrix}$ point, at which the map (†) undergoes a Hopf bifurcation.

[End of Paper]