M4 Dynamical Systems

May 2005 — Questions and Answers

2 hours; best 3 answers count.

1. Consider the two-dimensional system

$$x_{n+1} = -x_n + 3y_n - \frac{15}{8}(x_n - y_n)^3,$$

$$y_{n+1} = -\frac{3}{2}x_n + \frac{7}{2}y_n - \frac{15}{8}(x_n - y_n)^3,$$

where $x_i, y_i \in \mathbb{R}$.

(a) Show that there is a saddle-point at the origin.

(b) Give the definitions of the stable and unstable subspaces of the origin and find their equations.

(c) Introduce the vector $\begin{pmatrix} u_n^+ \\ u_n^- \end{pmatrix}$ which is defined via

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} \begin{pmatrix} u_n^+ \\ u_n^- \end{pmatrix},$$

where $\begin{pmatrix} a \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ b \end{pmatrix}$ are vectors aligned with the stable and unstable subspaces, respectively. Thereby, show that the system may be expressed in the form

$$u_{n+1}^+ = \alpha\, u_n^+,$$

$$u_{n+1}^- = \beta\, u_n^- + \gamma \left(u_n^+\right)^3;$$

and evaluate the constants a, b, α, β and γ.

(d) State the stable manifold theorem and show that

  i. the stable manifold is given *exactly* by

$$u^- = \delta \left(u^+\right)^3;$$

  ii. the unstable manifold is given *exactly* by

$$u^+ = \rho;$$

  and evaluate the constants δ and ρ.

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(e) Sketch the stable and unstable manifolds in the (u^+, u^-) plane. Include in your sketch a few representative orbits and indentify the stable and unstable subspaces.

Solution.

(a) We have:
$$\left.\begin{array}{l} x_{n+1} = -x_n + 3y_n - \frac{15}{8}(x_n - y_n)^3 \\ y_{n+1} = -\frac{3}{2}x_n + \frac{7}{2}y_n - \frac{15}{8}(x_n - y_n)^3 \end{array}\right\}$$

$$\Rightarrow \quad \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mathbf{F}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} -x_n + 3y_n - \frac{15}{8}(x_n - y_n)^3 \\ -\frac{3}{2}x_n + \frac{7}{2}y_n - \frac{15}{8}(x_n - y_n)^3 \end{pmatrix}$$

Jacobian of $\mathbf{F} = D\mathbf{F} = \begin{pmatrix} -1 - \frac{45}{8}(x-y)^2 & 3 + \frac{45}{8}(x-y)^2 \\ -\frac{3}{2} - \frac{45}{8}(x-y)^2 & \frac{7}{2} + \frac{45}{8}(x-y)^2 \end{pmatrix}$

$$\Rightarrow \quad D\mathbf{F}|_{(0,0)} = \begin{pmatrix} -1 & 3 \\ -\frac{3}{2} & \frac{7}{2} \end{pmatrix}.$$

Eigenvalues of $D\mathbf{F}|_{(0,0)}$:

$$(-1-\lambda)\left(\frac{7}{2}-\lambda\right) + \frac{9}{2} = 0$$

$$\Rightarrow \quad \lambda^2 - \frac{5}{2}\lambda + 2 = 0$$

$$\Rightarrow \quad (2\lambda - 1)(\lambda - 2) = 0 \quad \Rightarrow \quad \lambda_a = \frac{1}{2}, \quad \lambda_b = 2$$

Since $|\lambda_a| < 1$ and $|\lambda_b| > 1$, there is a saddle-point at the origin.

(b) The stable subspace E^+ is the span of eigenvectors associated with eigenvectors λ with $|\lambda| < 1$. The unstable subspace E^- is the span of eigenvectors associated with eigenvectors λ with $|\lambda| > 1$.

Eigenvectors of of $D\mathbf{F}|_{(0,0)}$:

$$\begin{pmatrix} -1 & 3 \\ -\frac{3}{2} & \frac{7}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad \mathbf{e}_a = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \text{stable subspace:} \quad y = \frac{1}{2}x$$

$$\begin{pmatrix} -1 & 3 \\ -\frac{3}{2} & \frac{7}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad \mathbf{e}_b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \text{unstable subspace:} \quad y = x$$

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(c)

$$\text{Diagonalising matrix } S = \begin{pmatrix} \mathbf{e}_a & \mathbf{e}_b \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

i.e., $\begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

i.e., $\left. \begin{aligned} u_n^+ &= x_n - y_n \\ u_n^- &= -x_n + 2y_n \end{aligned} \right\}$ and $\left. \begin{aligned} x_n &= 2u_n^+ + u_n^- \\ y_n &= u_n^+ + u_n^- \end{aligned} \right\}$

Rewriting system in terms of u^+ and u^-:

$$\left. \begin{aligned} 2u_{n+1}^+ + u_{n+1}^- &= -2u_n^+ - u_n^- + 3u_n^+ + 3u_n^- - \tfrac{15}{8}\left(u_n^+\right)^3 \\ &= u_n^+ + 2u_n^- - \tfrac{15}{8}\left(u_n^+\right)^3 \qquad (1) \\ u_{n+1}^+ + u_{n+1}^- &= -3u_n^+ - \tfrac{3}{2}u_n^- + \tfrac{7}{2}u_n^+ + \tfrac{7}{2}u_n^- - \tfrac{15}{8}\left(u_n^+\right)^3 \\ &= \tfrac{1}{2}u_n^+ + 2u_n^- - \tfrac{15}{8}\left(u_n^+\right)^3 \qquad (2) \end{aligned} \right\}$$

$(1) - (2)$ gives:

$$u_{n+1}^+ = \frac{1}{2}u_n^+$$

$[2 \times (2)] - (1)$ gives:

$$u_{n+1}^- = 2u_n^- - \frac{15}{8}\left(u_n^+\right)^3$$

(d) The stable manifold theorem asserts the existence of (i) an invariant stable manifold W^+ defined by $\left\{ \mathbf{x} : \mathbf{F}^k(\mathbf{x}) \to \mathbf{0} \quad \text{as} \quad k \to \infty \right\}$; and (ii) an invariant unstable manifold W^- defined by $\left\{ \mathbf{x} : \mathbf{F}^k(\mathbf{x}) \to \mathbf{0} \quad \text{as} \quad k \to -\infty \right\}$.

i. The stable manifold W^+ is expanded as

$$\begin{aligned} u_n^- &= a_2\left(u_n^+\right)^2 + a_3\left(u_n^+\right)^3 + \cdots \\ \Rightarrow \quad u_{n+1}^- &= a_2\left(u_{n+1}^+\right)^2 + a_3\left(u_{n+1}^+\right)^3 + \cdots \\ &= a_2\left(\frac{1}{2}u_n^+\right)^2 + a_3\left(\frac{1}{2}u_n^+\right)^3 + \text{cubic and higher order terms} \\ &= 2u_n^- - \frac{15}{8}\left(u_n^+\right)^3 \\ &= 2\left[a_2\left(u_n^+\right)^2 + a_3\left(u_n^+\right)^3 + \cdots\right] - \frac{15}{8}\left(u_n^+\right)^3 \end{aligned}$$

Comparing the coefficients of $\left(u_n^+\right)^2$ gives:

$$\frac{1}{4}a_2 = 2a_2 \qquad \Rightarrow \qquad a_2 = 0$$

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Comparing the coefficients of $(u_n^+)^3$ gives:

$$\frac{1}{8}a_3 = 2a_3 - \frac{15}{8} \qquad \Rightarrow \qquad a_3 = 1$$

And $a_k =$ for $k > 3$.

i.e., stable manifold is $u^- = (u^+)^3 + O((u^+)^4)$.

   ii. The unstable manifold W^- is, by inspection (or using the same method as for the stable manifold), is given by $u^+ = 0$.

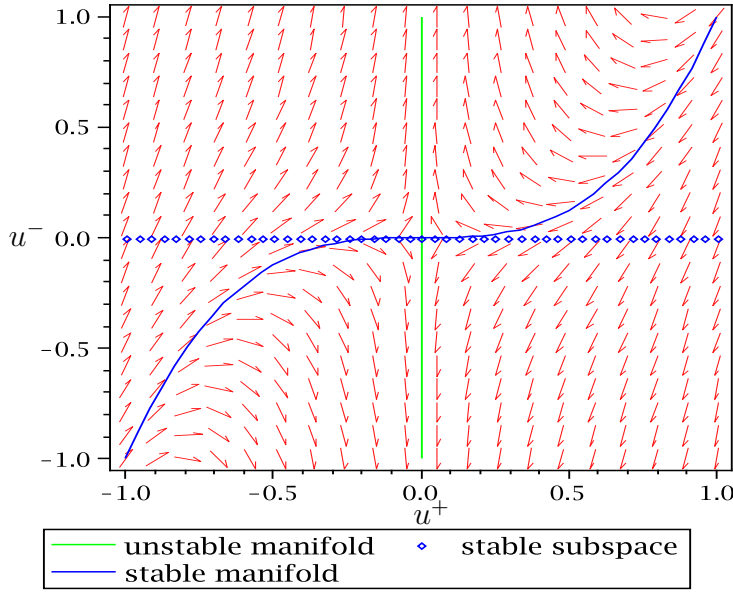(e) Sketch of stable and unstable manifolds:

Figure 1: Sketch of stable and unstable manifolds.

(f) $u_{n+1}^+ = \frac{1}{2}u_n^+ \qquad \Rightarrow \qquad p_{n+1} = \frac{1}{2}p_n$

We have $q_n = (u_n^+)^3 - u_n^- = (p_n)^3 - u_n^-$, so $u_{n+1}^- = 2u_n^- - \frac{15}{8}(u_n^+)^3$ becomes:

$$(p_{n+1})^3 - q_{n+1} = 2(p_n)^3 - 2q_n - \frac{15}{8}(p_n)^3$$

$$\text{i.e.,} \quad \frac{1}{8}(p_n)^3 - q_{n+1} = 2(p_n)^3 - 2q_n - \frac{15}{8}(p_n)^3$$

$$\Rightarrow \quad q_{n+1} = 2q_n$$

2. Consider the map

$$x_{n+2} = x_{n+1} - x_n + 2(2x_n - x_{n+1})^3, \tag{1}$$

where $x_i \in \mathbb{R}$.

(a) Find the fixed points of this map.

(b) Use the corresponding linearized map to discuss the stability of the fixed points.

(c) Let $z_n = x_n + \epsilon x_{n+1}$ where $\epsilon \in \mathbb{C}$. Show that, by choosing ϵ appropriately, the system (1) may expressed as

$$z_{n+1} = \alpha\, z_n + \beta_1\, (z_n)^3 + \beta_2\, (z_n)^2\, \bar{z}_n + \beta_3\, z_n\, (\bar{z}_n)^2 + \beta_4\, (\bar{z}_n)^3$$

and calculate the complex-valued constants α, β_1, β_2, β_3 and β_4.

(d) Explain briefly (without performing any calculations) how the system may rewritten in terms of a new variable ζ_n as

$$\zeta_{n+1} = \alpha\zeta_n + b\,\zeta_n^2\bar{\zeta}_n + O(|\zeta_n|^4)$$

and express b in terms of ϵ.

(e) Using (d), show how the stability of the origin depends on b/ϵ and thereby comment upon the stability of the origin for the system (1).

Solution.

(a) Fixed points $x = x^*$ satisfy

$$x^* = x^* - x^* + 2\left(2x^* - x^*\right)^3 ; \qquad \text{i.e.,} \quad x^* = 2\left(x^*\right)^3 ;$$

hence, there are 3 fixed points, namely $x^* = 0$, $x^* = 1/\sqrt{2}$ and $x^* = -1/\sqrt{2}$.

(b) By introducing the new variable $y_n = x_{n+1}$, the recurrence relation may be expressed as the $\mathbb{R}^2$ system

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mathbf{F} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} y_n \\ -x_n + y_n + 2\left(2x_n - y_n\right)^3 \end{pmatrix} ; \tag{1}$$

the three fixed points in terms of (x, y) coordinates are $(0, 0)$, $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. The Jacobian matrix of $\mathbf{F}$ is given by

$$D\mathbf{F} = \begin{pmatrix} 0 & 1 \\ -1 + 12\left(2x_n - y_n\right)^2 & 1 - 6\left(2x_n - y_n\right)^2 \end{pmatrix}.$$

The eigenvalues of $D\mathbf{F}|_{(0,0)}$ are given by the characteristic equation

$$(0 - \lambda_0)(1 - \lambda_0) + 1 = 0;$$

$$\lambda^2 - \lambda + 1 = 0 \qquad \text{i.e.,} \quad \lambda_0 = \frac{1 \pm i\sqrt{3}}{2};$$

since $|\lambda_0| = 1$, no conclusions can be drawn about the stability of the fixed point at $(0,0)$ using linear analysis. The eigenvalues of $D\mathbf{F}|_{(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})}$ are given by the characteristic equation

$$\left(0 - \lambda_{\pm 1/\sqrt{2}}\right)\left(-2 - \lambda_{\pm 1/\sqrt{2}}\right) - 5 = 0; \qquad \text{i.e.,} \quad \lambda_{\pm 1/\sqrt{2}} = -1 \pm \sqrt{6};$$

since $|\lambda_{\pm 1/\sqrt{2}}| > 1$, we deduce that the fixed points at $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$ are unstable.

(c) Following the method described in section 3.3 of the lecture notes, we let $A = D\mathbf{F}|_{(0,0)}$. An eigenvector $\begin{pmatrix} e_x \\ e_y \end{pmatrix}$ of A^T, associated with the eigenvalue λ_0, is given via

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e_x \\ e_y \end{pmatrix} = \lambda_0 \begin{pmatrix} e_x \\ e_y \end{pmatrix} \qquad \longrightarrow \qquad \begin{pmatrix} e_x \\ e_y \end{pmatrix} = \begin{pmatrix} 1 \\ -\lambda_0 \end{pmatrix}.$$

Now, let $z = (e_x, e_y) \cdot (x, y) = (1, -\lambda_0) \cdot (x, y) = x - \lambda_0 y$. Then

$$
\begin{aligned}
z_{n+1} &= x_{n+1} - \lambda_0 y_{n+1} \\
&= y_n - \lambda_0 \left[-x_n + y_n + 2\left(2x_n - y_n\right)^3 \right] & \text{substituting from eqn (1)} \\
&= \lambda_0 x_n + (1 - \lambda_0) y_n - 2\lambda_0 \left(2x_n - y_n\right)^3 \\
&= \lambda_0 z_n - 2\lambda_0 \left(2x_n - y_n\right)^3 & \text{since the characteristic equation is } \lambda_0^2 - \lambda_0 + 1 = 0.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
z + \bar{z} &= x - \lambda_0 y + x - \overline{\lambda_0} y \\
&= 2x - \left(\lambda_0 + \overline{\lambda_0}\right) y \\
&= 2x - 2\,\mathrm{Re}\,\{\lambda_0\}\, y \\
&= 2x - y,
\end{aligned}
$$

so we have that

$$
\begin{aligned}
z_{n+1} &= \lambda_0 z_n - 2\lambda_0 \left(z_n + \bar{z}_n\right)^3 \\
&= \lambda_0 z_n - 2\lambda_0 \left[(z_n)^3 + 3\,(z_n)^2\,\bar{z}_n + 3\,z_n\,(\bar{z}_n)^2 + (\bar{z}_n)^3 \right]. \tag{2}
\end{aligned}
$$

(iv) By introducing the new variable $\zeta_n = z_n + p z_n^3 + q z_n^2 \bar{z}_n + r z_n \bar{z}_n^2 + s \bar{z}_n^3$, the system may be expressed in the form

$$\zeta_{n+1} = \lambda_0 \zeta_n + \text{cubic and higher order terms};$$

by judiciously choosing the values of p, q, r and s, the terms involving ζ_n^3, $\overline{\zeta}_n^2 \zeta_n$ and $\overline{\zeta}_n^3$ can be eliminated, leaving

$$\zeta_{n+1} = \lambda_0 \zeta_n + b \zeta_n^2 \overline{\zeta}_n + \text{higher order terms}.$$

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The coefficient $\zeta_n^2 \bar{\zeta}_n$ is the same as the coefficient of $(z_n)^2 \bar{z}_n$ in eqn (2); i.e., $b = -6\lambda_0$.

(v) We have

$$
\begin{aligned}
|\zeta_{n+1}|^2 &= \left(\lambda_0 \zeta_n + b\zeta_n^2 \bar{\zeta}_n\right)\left(\bar{\lambda}_0 \bar{\zeta}_n + \bar{b}\bar{\zeta}_n^2 \zeta_n\right) + \text{higher order terms} \\
&= |\lambda_0|^2 |\zeta_n|^2 + \left(\lambda_0 \bar{b} + b\bar{\lambda}_0\right)|\zeta_n|^4 + \text{higher order terms} \\
&= |\zeta_n|^2 + \frac{1}{2}\operatorname{Re}\left\{\frac{b}{\lambda_0}\right\}|\zeta_n|^4 + \text{higher order terms.}
\end{aligned}
$$

So $|\zeta_n| \to 0$ if $\operatorname{Re}\left\{\frac{b}{\lambda_0}\right\} < 0$; i.e., the fixed point at the origin is stable provided that $\operatorname{Re}\left\{\frac{b}{\lambda_0}\right\} < 0$. Since we have here that $b = -6\lambda_0$, the fixed point at the origin is stable.

3. Consider the map
$$
H_\mu(x) = \mu \tan^{-1} x \,,
$$
where x is a real-valued variable and μ is a real-valued parameter.

(a) How many fixed points are there? Specify the ranges of values of μ for which they exist.

(b) Calculate the Schwarzian derivative of H_μ.

(c) Describe the bifurcations which occur for

    i. $\mu = 1$,

    ii. $\mu = -1$.

If there are flip bifurcations, state whether they are supercritical or subcritical. You may wish to make use of the following Taylor series expansion:

$$
\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \,,
$$

for $|x| < 1$.

(d) Sketch the bifurcation diagram in the (μ, x) plane; indicate the stability of the fixed points in your diagram.

Solution.

(a) Fixed points $x = x^*$ satisfy the equation $x^* = \mu \tan^{-1}(x^*)$. Clearly, there is a fixed point at $x_a^* = 0$. Notice that $\tan^{-1} x$ and x have the same sign, so there are no nonzeros solutions of $x^* = \mu \tan^{-1}(x^*)$ if $\mu < 0$. Also, the gradient of H_μ is given by
$$
H_\mu'(x) = \frac{\mu}{1 + x^2},
$$

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and $H'_\mu(0) = \mu$ Thus, for $\mu > 0$, $H'_\mu(x)$ is strictly increasing for $x \in (-\infty, 0)$ and strictly decreasing for $x \in (0, \infty)$. Hence, the line $y = x$ intersects the graph $y = H_\mu(x)$ at two further fixed points, namely x^*_b and $x^*_c = -x^*_b$, provided that $\mu > 1$.

(b) The Schwarzian derivative is defined as

$$D_S\{H_\mu\} = \frac{H'''_\mu}{H'_\mu} - \frac{3}{2}\left(\frac{H''_\mu}{H'_\mu}\right)^2$$

and since

$$H'_\mu(x) = \frac{\mu}{1+x^2}, \qquad H''_\mu(x) = -\frac{2\mu x}{(1+x^2)^2}, \qquad H'''_\mu(x) = \frac{8\mu x^2}{(1+x^2)^3} - \frac{2\mu}{(1+x^2)^2},$$

we have

$$\begin{aligned}
D_S\{H_\mu\} &= \frac{8x^2}{(1+x^2)^2} - \frac{2}{(1+x^2)} - \frac{3}{2}\left(-\frac{2x}{(1+x^2)}\right)^2 \\
&= \frac{8x^2 - 2(1+x^2) - 6x^2}{(1+x^2)^2} = -\frac{2}{(1+x^2)^2}
\end{aligned}$$

(c) Consider the bifurcation at $\mu = 1$. Let us introduce ϵ where $0 < \epsilon \ll 1$. For $\mu \in (1-\epsilon, 1)$, only the fixed point $x^*_a = 0$ exists; for $\mu \in (1, 1+\epsilon)$, the fixed points x^*_a, x^*_b and x^*_c all exist. The stability of fixed points is determined by $H'_\mu(x) = \frac{\mu}{1+x^2}$. Clearly, the fixed point $x^*_a = 0$ is stable for $\mu \in (1-\epsilon, 1)$ but unstable for $\mu \in (1, 1+\epsilon)$. Now, since fixed points satisfy $x^* = \mu \tan^{-1}(x^*)$, we have

$$H'_\mu(x^*) = \frac{x^*}{(1+(x^*)^2)\tan^{-1}x^*}.$$

For $\mu \in (1, 1+\epsilon)$, we have $x^*_b = \delta$ and $x^*_c = -\delta$, say, where $0 < \delta \ll 1$. Now, recalling the Taylor series $\tan^{-1}\delta = \delta - \frac{\delta^3}{3} + \cdots$, we have

$$\left.\begin{aligned}
H'_\mu(\delta) &= \frac{\delta}{(1+\delta^2)\tan^{-1}(\delta)} \approx \frac{\delta}{(1+\delta^2)\left(\delta - \frac{\delta^3}{3}\right)} \approx \frac{1}{1+\frac{2\delta^2}{3}} < 1 \\
H'_\mu(-\delta) &= \frac{-\delta}{(1+(-\delta)^2)\tan^{-1}(-\delta)} \approx \frac{\delta}{(1+\delta^2)\left(\delta - \frac{\delta^3}{3}\right)} \approx \frac{1}{1+\frac{2\delta^2}{3}} < 1
\end{aligned}\right\}.$$

Hence, the fixed points x^*_b and x^*_c are stable for $\mu \in (1, 1+\epsilon)$. Therefore, the bifurcation at $\mu = 1$ is of the pitchfork type.

Consider the bifurcation at $\mu = -1$. In the vicinity of $\mu = -1$, only the fixed point $x^*_a = 0$ exists. Since $H'_\mu(x) = \frac{\mu}{1+x^2}$, we see that the fixed point x^*_a, which is stable for $\mu > -1$, becomes unstable for $\mu < -1$; and at $\mu = -1$ we have $H'_\mu(x^*_a) = -1$. Thus, we have a flip birfurcation. Since the Schwarzian derivative is negative-valued (for all x), it is a supercritical flip bifurcation.
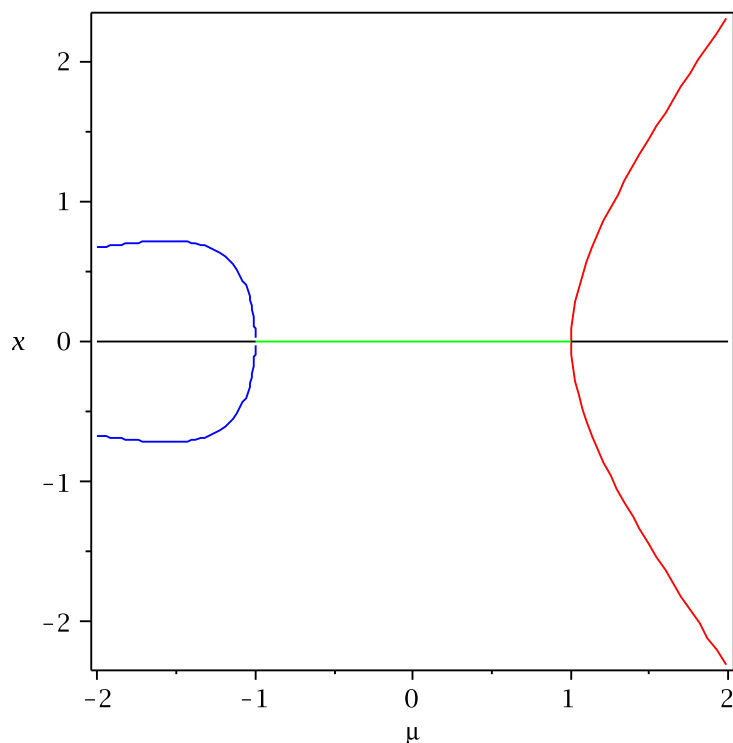
Figure 2: Legend: black is unstable fixed point at $x = 0$; green is stable fixed point at $x = 0$; red is pair of stable fixed points; blue is pair of stable period 2 points.

(d) Sketch: See figure 2.

4. (a) Suppose a continuous mapping G has a period-3 orbit (a, b, c) where $a < b < c$.
   i. Show that G has orbits of period 1 and period 2.
   ii. Show that G has orbits of *prime* period n for all $n > 3$.

   (b) Consider the $\mathbb{R}^2$ map represented as

   $$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n) \tag{2}$$

   where

   $$\mathbf{x}_{n+1} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}, \qquad \mathbf{x}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \qquad \mathbf{F}(\mathbf{x}_n) = \begin{pmatrix} y_n + \nu \\ y_n^2 - x_n^2 \end{pmatrix},$$

   and ν is a real-valued parameter.

   i. Find the $\begin{pmatrix} x \\ y \end{pmatrix}$ fixed points of the map (2) in terms of ν.

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ii. If the map (2) undergoes a Hopf bifurcation, what can you infer about the eigenvalues of the Jacobian matrix of derivatives of $\mathbf{F}$?

iii. Find the value of ν, and the corresponding $\begin{pmatrix} x \\ y \end{pmatrix}$ point, at which the map (2) undergoes a Hopf bifurcation.

Solution.

(a) Let us define $I_0 = [a, b]$ and $I_1 = [b, c]$ and make the following observations

    i. $G(I_0) \supseteq I_1$.

    ii. $G(I_1) \supseteq I_0 \cup I_1$.

    iii. If I is a closed interval and $G(I) \supseteq I$, then G has a fixed point in I.

    iv. Suppose I, J are closed intervals. If $G(I) \supseteq J$, then there exists a closed interval $K \subseteq I$ such that $G(K) = J$.

The last two observations can be established using the intermediate value theorem.

- We start by noting that (ii) and (iii) imply that G has a fixed point in I_1. Also, (i), (ii) and (iii) imply that G^2 has a fixed point in I_0, so that G has a period-2 orbit. Thus, the period-1 and period-2 cases are proven.

- We have $n > 3$. Now we construct a nested sequence of closed intervals A_n: let $A_0 = I_1$, (ii) and (iv) imply that there is a $A_1 \subseteq A_0$ with $G(A_1) = A_0 = I_1$. Similarly, there is a $A_2 \subseteq A_1$ with $G(A_2) = A_1$ and so $G^2(A_2) = A_0$. Proceeding similarly, the sequence

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_{n-2}, \quad \text{with} \quad G^k(A_k) = A_0, \quad k = 1, 2, \ldots, n-2,$$

can be constructed. The next interval in the sequence, A_{n-1} is constructed by noting that $G^{n-1}(A_{n-2}) = G(A_0) \supseteq I_0$ (using (ii)). Then, (iv) implies that there is a $A_{n-1} \subseteq A_{n-2}$ with $G^{n-1}(A_{n-1}) = I_0$. Finally since $G^n(A_{n-1}) = G(I_0) \supseteq I_1$ (using (i)), there exists a $A_n \subseteq A_{n-1}$ with $G^n(A_n) = A_0 = I_1$. Now, by construction $A_n \subseteq A_0$, so that $G^n(A_n) \supseteq A_n$. So (iii) then implies that there exists a fixed point $x^\star \in A_n$ with $G^n(x^\star) = x^\star$. This is a prime period-n point unless it is also fixed point of G^k for $k < n$. But this is impossible since $x^\star \in A_k$, $k = 0, 1, \cdots, n$ gives that $G^k(x^\star) \in I_1$ for $k = 1, 2, \ldots, n-2$ and we also have $G^{n-1}(x^\star) \in I_0$. (The case $G^{n-1}(x^\star) \in I_0 \cap I_1 = \{b\}$ can be excluded since it would imply $n = 3$.)

(b)   i. Fixed points satisfy

$$\begin{aligned} x = y + \nu \qquad &\text{and} \qquad y = y^2 - x^2 \\ \Rightarrow \qquad y &= y^2 - (y + \nu)^2 \\ \text{i.e.,} \qquad y &= -2\nu y - \nu^2 \\ \text{i.e.,} \qquad y &= -\frac{\nu^2}{1 + 2\nu} \qquad\qquad (\dagger). \end{aligned}$$

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ii. $\lambda_1 = \exp(i\sigma)$ and $\lambda_2 = \exp(-i\sigma)$ where $\sigma \neq 0, \pi$.

iii. The corresponding Jacobian matrix is $\begin{pmatrix} 0 & 1 \\ -2x & 2y \end{pmatrix}$; the eigenvalues satisfy

$$(0 - \lambda)(2y - \lambda) + 2x = 0 \quad \text{i.e.,} \quad \lambda^2 - 2y\lambda + 2x = 0$$

$$\text{i.e.,} \quad \lambda = \frac{2y \pm \sqrt{4y^2 - 8x}}{2} = y \pm \sqrt{y^2 - 2x}.$$

For Hopf bifurcation $\lambda = e^{i\sigma}$ where $\sigma \neq 0, \pi$; i.e., $\lambda_1 = e^{i\sigma}$ and $\lambda_2 = e^{-i\sigma}$. Then

$$\lambda_1 \lambda_2 = 1 = \text{determinant of Jacobian} = 2x;$$

$$\text{i.e.,} \quad x = \frac{1}{2}, \quad y = \frac{1}{2} - \nu.$$

Now, from (†), we have

$$\frac{1}{2} - \nu = -\frac{\nu^2}{1 + 2\nu} \quad \text{i.e.,} \quad \frac{1}{2} + \nu - \nu - 2\nu^2 = -\nu^2$$

$$\text{i.e.,} \quad \nu^2 = \frac{1}{2} \quad \text{and so} \quad \nu = \pm\frac{1}{\sqrt{2}}$$

For $\nu = -\frac{1}{\sqrt{2}}$, we have $x = \frac{1}{2}$, $y = \frac{1}{2} + \frac{1}{\sqrt{2}}$ and so $\lambda \in \mathbb{R}$.

For $\nu = \frac{1}{\sqrt{2}}$, we have $x = \frac{1}{2}$, $y = \frac{1}{2} - \frac{1}{\sqrt{2}}$ and so $\lambda = e^{\pm i\sigma}$ where $\sigma \neq 0, \pi$.

i.e., Hopf bifurcation arises for $\nu = -\frac{1}{\sqrt{2}}$ at $x = \frac{1}{2}$, $y = \frac{1}{2} - \frac{1}{\sqrt{2}}$.