M4 Dynamical Systems

May 2005 — Questions and Answers

2 hours; best 3 answers count.

1. Consider the two-dimensional system

$$x_{n+1} = -x_n + 3y_n - \frac{15}{8}(x_n - y_n)^3,$$
$$y_{n+1} = -\frac{3}{2}x_n + \frac{7}{2}y_n - \frac{15}{8}(x_n - y_n)^3,$$

where $x_i, y_i \in \mathbb{R}$.

(a) Show that there is a saddle-point at the origin.

(b) Give the definitions of the stable and unstable subspaces of the origin and find their equations.

(c) Introduce the vector $\begin{pmatrix} u_n^+ \\ u_n^- \end{pmatrix}$ which is defined via

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} \begin{pmatrix} u_n^+ \\ u_n^- \end{pmatrix},$$

where $\begin{pmatrix} a \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ b \end{pmatrix}$ are vectors aligned with the stable and unstable subspaces, respectively. Thereby, show that the system may be expressed in the form

$$u_{n+1}^+ = \alpha\, u_n^+,$$
$$u_{n+1}^- = \beta\, u_n^- + \gamma\, \left(u_n^+\right)^3;$$

and evaluate the constants a, b, α, β and γ.

(d) State the stable manifold theorem and show that

i. the stable manifold is given *exactly* by

$$u^- = \delta\, \left(u^+\right)^3;$$

ii. the unstable manifold is given *exactly* by

$$u^+ = \rho;$$

and evaluate the constants δ and ρ.

(e) Sketch the stable and unstable manifolds in the (u^+, u^-) plane. Include in your sketch a few representative orbits and indentify the stable and unstable subspaces.

Solution.

(a) We have:
$$\left.\begin{aligned} x_{n+1} &= -x_n + 3y_n - \tfrac{15}{8}(x_n - y_n)^3 \\ y_{n+1} &= -\tfrac{3}{2}x_n + \tfrac{7}{2}y_n - \tfrac{15}{8}(x_n - y_n)^3 \end{aligned}\right\}$$

$$\Rightarrow \quad \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mathbf{F}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} -x_n + 3y_n - \frac{15}{8}(x_n - y_n)^3 \\ -\frac{3}{2}x_n + \frac{7}{2}y_n - \frac{15}{8}(x_n - y_n)^3 \end{pmatrix}$$

Jacobian of $\mathbf{F} = D\mathbf{F} = \begin{pmatrix} -1 - \frac{45}{8}(x-y)^2 & 3 + \frac{45}{8}(x-y)^2 \\ -\frac{3}{2} - \frac{45}{8}(x-y)^2 & \frac{7}{2} + \frac{45}{8}(x-y)^2 \end{pmatrix}$

$\Rightarrow \quad D\mathbf{F}|_{(0,0)} = \begin{pmatrix} -1 & 3 \\ -\frac{3}{2} & \frac{7}{2} \end{pmatrix}.$

Eigenvalues of $D\mathbf{F}|_{(0,0)}$:

$$(-1-\lambda)\left(\frac{7}{2} - \lambda\right) + \frac{9}{2} = 0$$
$$\Rightarrow \quad \lambda^2 - \frac{5}{2}\lambda + 2 = 0$$
$$\Rightarrow \quad (2\lambda - 1)(\lambda - 2) = 0 \quad \Rightarrow \quad \lambda_a = \frac{1}{2}, \quad \lambda_b = 2$$

Since $|\lambda_a| < 1$ and $|\lambda_b| > 1$, there is a saddle-point at the origin.

(b) The stable subspace E^+ is the span of eigenvectors associated with eigenvectors λ with $|\lambda| < 1$. The unstable subspace E^- is the span of eigenvectors associated with eigenvectors λ with $|\lambda| > 1$.

Eigenvectors of of $D\mathbf{F}|_{(0,0)}$:

$$\begin{pmatrix} -1 & 3 \\ -\frac{3}{2} & \frac{7}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad \mathbf{e}_a = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$\Rightarrow \quad \text{stable subspace:} \quad y = \frac{1}{2}x$$
$$\begin{pmatrix} -1 & 3 \\ -\frac{3}{2} & \frac{7}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \quad \Rightarrow \quad \mathbf{e}_b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\Rightarrow \quad \text{unstable subspace:} \quad y = x$$

(c)

$$\text{Diagonalising matrix } S = \begin{pmatrix} \mathbf{e}_a & \mathbf{e}_b \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\text{i.e.,} \quad \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\text{i.e.,} \quad \left.\begin{array}{l} u_n^+ = x_n - y_n \\ u_n^- = -x_n + 2y_n \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l} x_n = 2u_n^+ + u_n^- \\ y_n = u_n^+ + u_n^- \end{array}\right\}$$

Rewriting system in terms of u^+ and u^-:

$$\left.\begin{array}{rcll} 2u_{n+1}^+ + u_{n+1}^- &=& -2u_n^+ - u_n^- + 3u_n^+ + 3u_n^- - \frac{15}{8}\left(u_n^+\right)^3 & \\ &=& u_n^+ + 2u_n^- - \frac{15}{8}\left(u_n^+\right)^3 & (1) \\ u_{n+1}^+ + u_{n+1}^- &=& -3u_n^+ - \frac{3}{2}u_n^- + \frac{7}{2}u_n^+ + \frac{7}{2}u_n^- - \frac{15}{8}\left(u_n^+\right)^3 & \\ &=& \frac{1}{2}u_n^+ + 2u_n^- - \frac{15}{8}\left(u_n^+\right)^3 & (2) \end{array}\right\}$$

(1) − (2) gives:

$$u_{n+1}^+ = \frac{1}{2}u_n^+$$

[2 × (2)] − (1) gives:

$$u_{n+1}^- = 2u_n^- - \frac{15}{8}\left(u_n^+\right)^3$$

(d) The stable manifold theorem asserts the existence of (i) an invariant stable manifold W^+ defined by $\left\{\mathbf{x} : \mathbf{F}^k(\mathbf{x}) \to \mathbf{0} \quad \text{as} \quad k \to \infty\right\}$; and (ii) an invariant unstable manifold W^- defined by $\left\{\mathbf{x} : \mathbf{F}^k(\mathbf{x}) \to \mathbf{0} \quad \text{as} \quad k \to -\infty\right\}$.

i. The stable manifold W^+ is expanded as

$$\begin{array}{rrcl} & u_n^- &=& a_2\left(u_n^+\right)^2 + a_3\left(u_n^+\right)^3 + \cdots \\ \Rightarrow & u_{n+1}^- &=& a_2\left(u_{n+1}^+\right)^2 + a_3\left(u_{n+1}^+\right)^3 + \cdots \\ & &=& a_2\left(\frac{1}{2}u_n^+\right)^2 + a_3\left(\frac{1}{2}u_n^+\right)^3 + \text{cubic and higher order terms} \\ & &=& 2u_n^- - \frac{15}{8}\left(u_n^+\right)^3 \\ & &=& 2\left[a_2\left(u_n^+\right)^2 + a_3\left(u_n^+\right)^3 + \cdots\right] - \frac{15}{8}\left(u_n^+\right)^3 \end{array}$$

Comparing the coefficients of $(u_n^+)^2$ gives:

$$\frac{1}{4}a_2 = 2a_2 \qquad \Rightarrow \qquad a_2 = 0$$

Comparing the coefficients of $(u_n^+)^3$ gives:

$$\frac{1}{8}a_3 = 2a_3 - \frac{15}{8} \qquad \Rightarrow \qquad a_3 = 1$$

And $a_k =$ for $k > 3$.
i.e., stable manifold is $u^- = (u^+)^3 + O((u^+)^4)$.

ii. The unstable manifold W^- is, by inspection (or using the same method as for the stable manifold), is given by $u^+ = 0$.

(e) Sketch of stable and unstable manifolds:

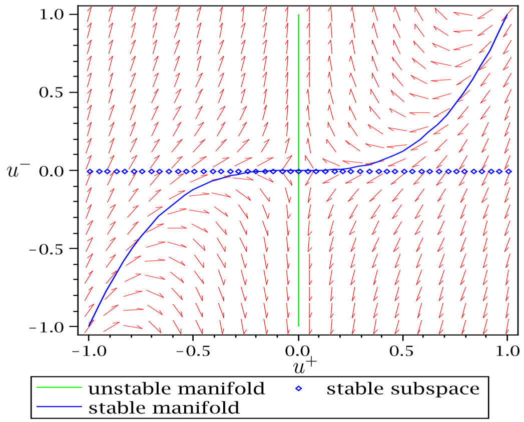

Figure 1: Sketch of stable and unstable manifolds.

(f) $u_{n+1}^+ = \frac{1}{2}u_n^+ \qquad \Rightarrow \qquad p_{n+1} = \frac{1}{2}p_n$
We have $q_n = (u_n^+)^3 - u_n^- = (p_n)^3 - u_n^-$, so $u_{n+1}^- = 2u_n^- - \frac{15}{8}\left(u_n^+\right)^3$ becomes:

$$\begin{array}{rrcl} & (p_{n+1})^3 - q_{n+1} &=& 2(p_n)^3 - 2q_n - \frac{15}{8}(p_n)^3 \\ \text{i.e.,} & \frac{1}{8}(p_n)^3 - q_{n+1} &=& 2(p_n)^3 - 2q_n - \frac{15}{8}(p_n)^3 \\ \Rightarrow & q_{n+1} &=& 2q_n \end{array}$$

2. Consider the map

$$x_{n+2} = x_{n+1} - x_n + 2(2x_n - x_{n+1})^3, \qquad (1)$$

where $x_i \in \mathbb{R}$.

(a) Find the fixed points of this map.

(b) Use the corresponding linearized map to discuss the stability of the fixed points.

(c) Let $z_n = x_n + \epsilon x_{n+1}$ where $\epsilon \in \mathbb{C}$. Show that, by choosing ϵ appropriately, the system (1) may expressed as

$$z_{n+1} = \alpha\, z_n + \beta_1\, (z_n)^3 + \beta_2\, (z_n)^2\, \overline{z}_n + \beta_3\, z_n\, (\overline{z}_n)^2 + \beta_4\, (\overline{z}_n)^3$$

and calculate the complex-valued constants α, β_1, β_2, β_3 and β_4.

(d) Explain briefly (without performing any calculations) how the system may rewritten in terms of a new variable ζ_n as

$$\zeta_{n+1} = \alpha\zeta_n + b\,\zeta_n^2\overline{\zeta}_n + O(|\zeta_n|^4)$$

and express b in terms of ϵ.

(e) Using (d), show how the stability of the origin depends on b/ϵ and thereby comment upon the stability of the origin for the system (1).

Solution.

(a) Fixed points $x = x^*$ satisfy

$$x^* = x^* - x^* + 2\,(2x^* - x^*)^3\ ; \qquad \text{i.e.,} \quad x^* = 2\,(x^*)^3\ ;$$

hence, there are 3 fixed points, namely $x^* = 0$, $x^* = 1/\sqrt{2}$ and $x^* = -1/\sqrt{2}$.

(b) By introducing the new variable $y_n = x_{n+1}$, the recurrence relation may be expressed as the $\mathbb{R}^2$ system

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mathbf{F}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} y_n \\ -x_n + y_n + 2\,(2x_n - y_n)^3 \end{pmatrix}; \qquad (1)$$

the three fixed points in terms of (x, y) coordinates are $(0, 0)$, $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. The Jacobian matrix of $\mathbf{F}$ is given by

$$D\mathbf{F} = \begin{pmatrix} 0 & 1 \\ -1 + 12\,(2x_n - y_n)^2 & 1 - 6\,(2x_n - y_n)^2 \end{pmatrix}.$$

The eigenvalues of $D\mathbf{F}|_{(0,0)}$ are given by the characteristic equation

$$(0 - \lambda_0)\,(1 - \lambda_0) + 1 = 0;$$
$$\lambda^2 - \lambda + 1 = 0 \qquad \text{i.e.,} \quad \lambda_0 = \frac{1 \pm i\sqrt{3}}{2};$$

since $|\lambda_0| = 1$, no conclusions can be drawn about the stability of the fixed point at $(0, 0)$ using linear analysis. The eigenvalues of $D\mathbf{F}|_{(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})}$ are given by the characteristic equation

$$\left(0 - \lambda_{\pm 1/\sqrt{2}}\right)\left(-2 - \lambda_{\pm 1/\sqrt{2}}\right) - 5 = 0; \qquad \text{i.e.,} \quad \lambda_{\pm 1/\sqrt{2}} = -1 \pm \sqrt{6};$$

since $|\lambda_{\pm 1/\sqrt{2}}| > 1$, we deduce that the fixed points at $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$ are unstable.

(c) Following the method described in section 3.3 of the lecture notes, we let $A = D\mathbf{F}|_{(0,0)}$. An eigenvector $\begin{pmatrix} e_x \\ e_y \end{pmatrix}$ of A^T, associated with the eigenvalue λ_0, is given via

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e_x \\ e_y \end{pmatrix} = \lambda_0 \begin{pmatrix} e_x \\ e_y \end{pmatrix} \qquad \longrightarrow \qquad \begin{pmatrix} e_x \\ e_y \end{pmatrix} = \begin{pmatrix} 1 \\ -\lambda_0 \end{pmatrix}.$$

Now, let $z = (e_x, e_y) \bullet (x, y) = (1, -\lambda_0) \bullet (x, y) = x - \lambda_0 y$. Then

$$\begin{aligned} z_{n+1} &= x_{n+1} - \lambda_0 y_{n+1} \\ &= y_n - \lambda_0 \left[-x_n + y_n + 2\,(2x_n - y_n)^3\right] && \text{substituting from eqn (1)} \\ &= \lambda_0 x_n + (1 - \lambda_0)\, y_n - 2\lambda_0\,(2x_n - y_n)^3 \\ &= \lambda_0 z_n - 2\lambda_0\,(2x_n - y_n)^3 && \text{since the characteristic equation is } \lambda_0^2 - \lambda_0 + 1 = 0. \end{aligned}$$

Notice that

$$\begin{aligned} z + \overline{z} &= x - \lambda_0 y + x - \overline{\lambda_0} y \\ &= 2x - \left(\lambda_0 + \overline{\lambda_0}\right) y \\ &= 2x - 2\,\mathrm{Re}\,\{\lambda_0\}\, y \\ &= 2x - y, \end{aligned}$$

so we have that

$$\begin{aligned} z_{n+1} &= \lambda_0 z_n - 2\lambda_0\,(z_n + \overline{z}_n)^3 \\ &= \lambda_0 z_n - 2\lambda_0 \left[(z_n)^3 + 3\,(z_n)^2\,\overline{z}_n + 3\,z_n\,(\overline{z}_n)^2 + (\overline{z}_n)^3\right]. \qquad (2) \end{aligned}$$

(iv) By introducing the new variable $\zeta_n = z_n + p z_n^3 + q z_n^2 \overline{z}_n + r z_n \overline{z}_n^2 + s\overline{z}_n^3$, the system may be expressed in the form

$$\zeta_{n+1} = \lambda_0 \zeta_n + \text{cubic and higher order terms;}$$

by judiciously choosing the values of p, q, r and s, the terms involving ζ_n^3, $\overline{\zeta}_n^2\zeta_n$ and $\overline{\zeta}_n^3$ can be eliminated, leaving

$$\zeta_{n+1} = \lambda_0\zeta_n + b\zeta_n^2\overline{\zeta}_n + \text{higher order terms.}$$

The coefficient $\zeta_n^2\overline{\zeta}_n$ is the same as the coefficient of $(z_n)^2\,\overline{z}_n$ in eqn (2); i.e., $b = -6\lambda_0$.

(v) We have

$$\begin{aligned}|\zeta_{n+1}|^2 &= \left(\lambda_0\zeta_n + b\zeta_n^2\overline{\zeta}_n\right)\left(\overline{\lambda}_0\overline{\zeta}_n + \overline{b}\overline{\zeta}_n^2\zeta_n\right) + \text{higher order terms}\\ &= |\lambda_0|^2|\zeta_n|^2 + \left(\lambda_0\overline{b} + b\overline{\lambda}_0\right)|\zeta_n|^4 + \text{higher order terms}\\ &= |\zeta_n|^2 + \frac{1}{2}\,\mathrm{Re}\left\{\frac{b}{\lambda_0}\right\}|\zeta_n|^4 + \text{higher order terms.}\end{aligned}$$

So $|\zeta_n| \to 0$ if $\mathrm{Re}\left\{\frac{b}{\lambda_0}\right\} < 0$; i.e., the fixed point at the origin is stable provided that $\mathrm{Re}\left\{\frac{b}{\lambda_0}\right\} < 0$. Since we have here that $b = -6\lambda_0$, the fixed point at the origin is stable.

3. Consider the map

$$H_\mu(x) = \mu\tan^{-1}x\,,$$

where x is a real-valued variable and μ is a real-valued parameter.

(a) How many fixed points are there? Specify the ranges of values of μ for which they exist.

(b) Calculate the Schwarzian derivative of H_μ.

(c) Describe the bifurcations which occur for

i. $\mu = 1$,

ii. $\mu = -1$.

If there are flip bifurcations, state whether they are supercritical or subcritical. You may wish to make use of the following Taylor series expansion:

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\,,$$

for $|x| < 1$.

(d) Sketch the bifurcation diagram in the (μ, x) plane; indicate the stability of the fixed points in your diagram.

Solution.

(a) Fixed points $x = x^*$ satisfy the equation $x^* = \mu\tan^{-1}(x^*)$. Clearly, there is a fixed point at $x_a^* = 0$. Notice that $\tan^{-1}x$ and x have the same sign, so there are no nonzeros solutions of $x^* = \mu\tan^{-1}(x^*)$ if $\mu < 0$. Also, the gradient of H_μ is given by

$$H'_\mu(x) = \frac{\mu}{1+x^2},$$

and $H'_\mu(0) = \mu$ Thus, for $\mu > 0$, $H'_\mu(x)$ is strictly increasing for $x \in (-\infty, 0)$ and strictly decreasing for $x \in (0, \infty)$. Hence, the line $y = x$ intersects the graph $y = H_\mu(x)$ at two further fixed points, namely x_b^* and $x_c^* = -x_b^*$, provided that $\mu > 1$.

(b) The Schwarzian derivative is defined as

$$D_S\{H_\mu\} = \frac{H'''_\mu}{H'_\mu} - \frac{3}{2}\left(\frac{H''_\mu}{H'_\mu}\right)^2$$

and since

$$H'_\mu(x) = \frac{\mu}{1+x^2}, \qquad H''_\mu(x) = -\frac{2\mu x}{(1+x^2)^2}, \qquad H'''_\mu(x) = \frac{8\mu x^2}{(1+x^2)^3} - \frac{2\mu}{(1+x^2)^2},$$

we have

$$\begin{aligned}D_S\{H_\mu\} &= \frac{8x^2}{(1+x^2)^2} - \frac{2}{(1+x^2)} - \frac{3}{2}\left(-\frac{2x}{(1+x^2)}\right)^2\\ &= \frac{8x^2 - 2(1+x^2) - 6x^2}{(1+x^2)^2} = -\frac{2}{(1+x^2)^2}\end{aligned}$$

(c) Consider the bifurcation at $\mu = 1$. Let us introduce ϵ where $0 < \epsilon \ll 1$. For $\mu \in (1-\epsilon, 1)$, only the fixed point $x_a^* = 0$ exists; for $\mu \in (1, 1+\epsilon)$, the fixed points x_a^*, x_b^* and x_c^* all exist. The stability of fixed points is determined by $H'_\mu(x) = \frac{\mu}{1+x^2}$. Clearly, the fixed point $x_a^* = 0$ is stable for $\mu \in (1-\epsilon, 1)$ but unstable for $\mu \in (1, 1+\epsilon)$. Now, since fixed points satisfy $x^* = \mu\tan^{-1}(x^*)$, we have

$$H'_\mu(x^*) = \frac{x^*}{(1+(x^*)^2)\tan^{-1}x^*}.$$

For $\mu \in (1, 1+\epsilon)$, we have $x_b^* = \delta$ and $x_c^* = -\delta$, say, where $0 < \delta \ll 1$. Now, recalling the Taylor series $\tan^{-1}\delta = \delta - \frac{\delta^3}{3} + \cdots$, we have

$$\left.\begin{aligned}H'_\mu(\delta) &= \tfrac{\delta}{(1+\delta^2)\tan^{-1}(\delta)} \approx \tfrac{\delta}{(1+\delta^2)\left(\delta-\frac{\delta^3}{3}\right)} \approx \tfrac{1}{1+\frac{2\delta^2}{3}} < 1\\ H'_\mu(-\delta) &= \tfrac{-\delta}{(1+(-\delta)^2)\tan^{-1}(-\delta)} \approx \tfrac{\delta}{(1+\delta^2)\left(\delta-\frac{\delta^3}{3}\right)} \approx \tfrac{1}{1+\frac{2\delta^2}{3}} < 1\end{aligned}\right\}.$$

Hence, the fixed points x_b^* and x_c^* are stable for $\mu \in (1, 1+\epsilon)$. Therefore, the bifurcation at $\mu = 1$ is of the pitchfork type.

Consider the bifurcation at $\mu = -1$. In the vicinity of $\mu = -1$, only the fixed point $x_a^* = 0$ exists. Since $H'_\mu(x) = \frac{\mu}{1+x^2}$, we see that the fixed point x_a^*, which is stable for $\mu > -1$, becomes unstable for $\mu < -1$; and at $\mu = -1$ we have $H'_\mu(x_a^*) = -1$. Thus, we have a flip birfurcation. Since the Schwarzian derivative is negative-valued (for all x), it is a supercritical flip bifurcation.

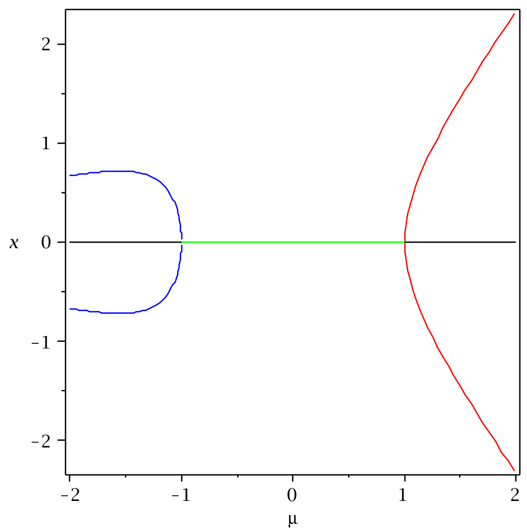

Figure 2: Legend: black is unstable fixed point at $x = 0$; green is stable fixed point at $x = 0$; red is pair of stable fixed points; blue is pair of stable period 2 points.

(d) Sketch: See figure 2.

4. (a) Suppose a continuous mapping G has a period-3 orbit (a, b, c) where $a < b < c$.
   i. Show that G has orbits of period 1 and period 2.
   ii. Show that G has orbits of *prime* period n for all $n > 3$.

   (b) Consider the $\mathbb{R}^2$ map represented as

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n) \qquad (2)$$

where

$$\mathbf{x}_{n+1} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}, \qquad \mathbf{x}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \qquad \mathbf{F}(\mathbf{x}_n) = \begin{pmatrix} y_n + \nu \\ y_n^2 - x_n^2 \end{pmatrix},$$

and ν is a real-valued parameter.

   i. Find the $\begin{pmatrix} x \\ y \end{pmatrix}$ fixed points of the map (2) in terms of ν.

   ii. If the map (2) undergoes a Hopf bifurcation, what can you infer about the eigenvalues of the Jacobian matrix of derivatives of $\mathbf{F}$?
   iii. Find the value of ν, and the corresponding $\begin{pmatrix} x \\ y \end{pmatrix}$ point, at which the map (2) undergoes a Hopf bifurcation.

Solution.

(a) Let us define $I_0 = [a, b]$ and $I_1 = [b, c]$ and make the following observations
   i. $G(I_0) \supseteq I_1$.
   ii. $G(I_1) \supseteq I_0 \cup I_1$.
   iii. If I is a closed interval and $G(I) \supseteq I$, then G has a fixed point in I.
   iv. Suppose I, J are closed intervals. If $G(I) \supseteq J$, then there exists a closed interval $K \subseteq I$ such that $G(K) = J$.

The last two observations can be established using the intermediate value theorem.

- We start by noting that (ii) and (iii) imply that G has a fixed point in I_1. Also, (i), (ii) and (iii) imply that G^2 has a fixed point in I_0, so that G has a period-2 orbit. Thus, the period-1 and period-2 cases are proven.
- We have $n > 3$. Now we construct a nested sequence of closed intervals A_n: let $A_0 = I_1$, (ii) and (iv) imply that there is a $A_1 \subseteq A_0$ with $G(A_1) = A_0 = I_1$. Similarly, there is a $A_2 \subseteq A_1$ with $G(A_2) = A_1$ and so $G^2(A_2) = A_0$. Proceeding similarly, the sequence

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_{n-2}, \quad \text{with} \quad G^k(A_k) = A_0, \quad k = 1, 2, \ldots, n-2,$$

can be constructed. The next interval in the sequence, A_{n-1} is constructed by noting that $G^{n-1}(A_{n-2}) = G(A_0) \supseteq I_0$ (using (ii)). Then, (iv) implies that there is a $A_{n-1} \subseteq A_{n-2}$ with $G^{n-1}(A_{n-1}) = I_0$. Finally since $G^n(A_{n-1}) = G(I_0) \supseteq I_1$ (using (i)), there exists a $A_n \subseteq A_{n-1}$ with $G^n(A_n) = A_0 = I_1$. Now, by construction $A_n \subseteq A_0$, so that $G^n(A_n) \supseteq A_n$. So (iii) then implies that there exists a fixed point $x^\star \in A_n$ with $G^n(x^\star) = x^\star$. This is a prime period-n point unless it is also fixed point of G^k for $k < n$. But this is impossible since $x^\star \in A_k$, $k = 0, 1, \cdots, n$ gives that $G^k(x^\star) \in I_1$ for $k = 1, 2, \ldots, n-2$ and we also have $G^{n-1}(x^\star) \in I_0$. (The case $G^{n-1}(x^\star) \in I_0 \cap I_1 = \{b\}$ can be excluded since it would imply $n = 3$.)

(b) i. Fixed points satisfy

$$x = y + \nu \qquad \text{and} \qquad y = y^2 - x^2$$
$$\Rightarrow \quad y = y^2 - (y+\nu)^2$$
$$\text{i.e.,} \quad y = -2\nu y - \nu^2$$
$$\text{i.e.,} \quad y = -\frac{\nu^2}{1+2\nu} \qquad (\dagger).$$

ii. $\lambda_1 = \exp(i\sigma)$ and $\lambda_2 = \exp(-i\sigma)$ where $\sigma \neq 0, \pi$.

iii. The corresponding Jacobian matrix is $\begin{pmatrix} 0 & 1 \\ -2x & 2y \end{pmatrix}$; the eigenvalues satisfy

$$\begin{aligned} (0-\lambda)(2y-\lambda)+2x &= 0 \qquad \text{i.e.,} \quad \lambda^2 - 2y\lambda + 2x = 0 \\ \text{i.e.,} \qquad \lambda &= \frac{2y \pm \sqrt{4y^2-8x}}{2} = y \pm \sqrt{y^2-2x}. \end{aligned}$$

For Hopf bifurcation $\lambda = e^{i\sigma}$ where $\sigma \neq 0, \pi$; i.e., $\lambda_1 = e^{i\sigma}$ and $\lambda_2 = e^{-i\sigma}$. Then

$$\begin{aligned} \lambda_1\lambda_2 &= 1 = \text{determinant of Jacobian} = 2x; \\ \text{i.e.,} \qquad & x = \frac{1}{2}, \qquad y = \frac{1}{2} - \nu\,. \end{aligned}$$

Now, from (†), we have

$$\begin{aligned} \frac{1}{2} - \nu &= -\frac{\nu^2}{1+2\nu} \qquad \text{i.e.,} \qquad \frac{1}{2} + \nu - \nu - 2\nu^2 = -\nu^2 \\ \text{i.e.,} \qquad \nu^2 &= \frac{1}{2} \qquad \text{and so} \qquad \nu = \pm\frac{1}{\sqrt{2}} \end{aligned}$$

For $\nu = -\frac{1}{\sqrt{2}}$, we have $x = \frac{1}{2}$, $y = \frac{1}{2} + \frac{1}{\sqrt{2}}$ and so $\lambda \in \mathbb{R}$.

For $\nu = \frac{1}{\sqrt{2}}$, we have $x = \frac{1}{2}$, $y = \frac{1}{2} - \frac{1}{\sqrt{2}}$ and so $\lambda = e^{\pm i\sigma}$ where $\sigma \neq 0, \pi$.

i.e., Hopf bifurcation arises for $\nu = -\frac{1}{\sqrt{2}}$ at $x = \frac{1}{2}$, $y = \frac{1}{2} - \frac{1}{\sqrt{2}}$.