Please hand in answers no later than **Friday 5th November**.

(**Question 1**) Find the fixed points of the dynamical system $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n)$ where

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{and} \qquad \mathbf{F}(\mathbf{x}) = \begin{pmatrix} x^2 \cos \pi y \\ y^3 \\ xz^2 + y \end{pmatrix},$$

and determine their stability.

(**Question 2**) Consider the relation

$$x_{n+1} = x_n - x_{n-1} + 2(2x_{n-1} - x_n)^3, \tag{0.1}$$

where $x_n \in \mathbb{R}$.

(a) Introduce $y_n = x_{n-1}$ and write this relation as a dynamical system $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$ where $\mathbf{f}$ is a map from $\mathbb{R}^2$ to itself.

(b) Find the fixed points of $\mathbf{f}$.

(c) Can the corresponding linearized map determine the stability of the fixed points?

(d) Let $z_n = x_{n-1} + \epsilon x_n$ for some $\epsilon \in \mathbb{C}$. Show that, by choosing ϵ appropriately, the system (0.1) may expressed as

$$z_{n+1} = \alpha\, z_n + \beta_1\, z_n^3 + \beta_2\, z_n^2 \overline{z}_n + \beta_3\, z_n \overline{z}_n^2 + \beta_4\, \overline{z}_n^3$$

and calculate the complex-valued constants α, β_1, β_2, β_3 and β_4.

(e) Explain briefly (without performing any calculations) how the system may rewritten in terms of a new variable ζ_n as

$$\zeta_{n+1} = \alpha \zeta_n + b\, \zeta_n^2 \overline{\zeta}_n + O(|\zeta_n|^4).$$

Determine b. [NB: it is not necessary to replicate the calculations in the notes. You may use the results.]

(f) Using (iv), show how the stability of the origin depends on b/ϵ and thereby comment upon the stability of the origin for the system (0.1).