Please hand in answers no later than **Tuesday 19 October**.

(**Question 1**) Consider the *linear* two-dimensional system

$$\left.\begin{aligned} x_{n+1} &= -x_n + 3y_n \\ y_{n+1} &= -\frac{3}{2}x_n + \frac{7}{2}y_n \end{aligned}\right\} \qquad (*),$$

where $x_n, y_n \in \mathbb{R}$.

(a) Show that there is a saddle-point at the origin.

(b) Find the equations of the stable and unstable subspaces at the origin.

(**Question 2**) Consider the *nonlinear* two-dimensional system

$$\left.\begin{aligned} x_{n+1} &= -x_n + 3y_n - \frac{15}{8}(x_n - y_n)^3 \\ y_{n+1} &= -\frac{3}{2}x_n + \frac{7}{2}y_n - \frac{15}{8}(x_n - y_n)^3 \end{aligned}\right\}, \qquad (**)$$

where $x_n, y_n \in \mathbb{R}$.

(a) Show that there is a saddle-point at the origin.

(b) Find the equations of the stable and unstable subspaces at the origin.

(c) Introduce the vector $\begin{pmatrix} u_n^+ \\ u_n^- \end{pmatrix}$ which is defined via

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} \begin{pmatrix} u_n^+ \\ u_n^- \end{pmatrix}, \qquad (\$)$$

where $\begin{pmatrix} a \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ b \end{pmatrix}$ are vectors aligned with the stable and unstable subspaces, respectively. Thereby, show that the nonlinear system may be expressed in the form

$$\left.\begin{aligned} u_{n+1}^+ &= \alpha\, u_n^+, \\ u_{n+1}^- &= \beta\, u_n^- + \gamma\, (u_n^+)^3 \end{aligned}\right\} \qquad (\$\$)$$

and evaluate the constants a, b, α, β and γ.

(d) Show that

i. the stable manifold is given *exactly* by

$$u^- = \delta\,\left(u^+\right)^3;$$

ii. the unstable manifold is given *exactly* by

$$u^+ = \rho;$$

and evaluate the constants δ and ρ.

(e) Sketch the stable and unstable manifolds in the (u^+, u^-) plane. Include in your sketch a few representative orbits and identify the stable and unstable subspaces.

(f) Show that the nonlinear system reduces to a linear system by the variable change

$$\begin{aligned} p_n &= u_n^+ \,, \\ q_n &= \delta \left(u_n^+\right)^3 - u_n^- \,. \end{aligned}$$

(**Question 3**) Let $I = [0, 1]$ and let $T : I \to I$ be the tent map defined by

$$T(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}] \\ -2(x-1) & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Prove that T has exactly 2^n periodic points of period n. Compute the number of prime periodic points of period n for $n = 6$.