Topology & Geometry of Integrable Systems

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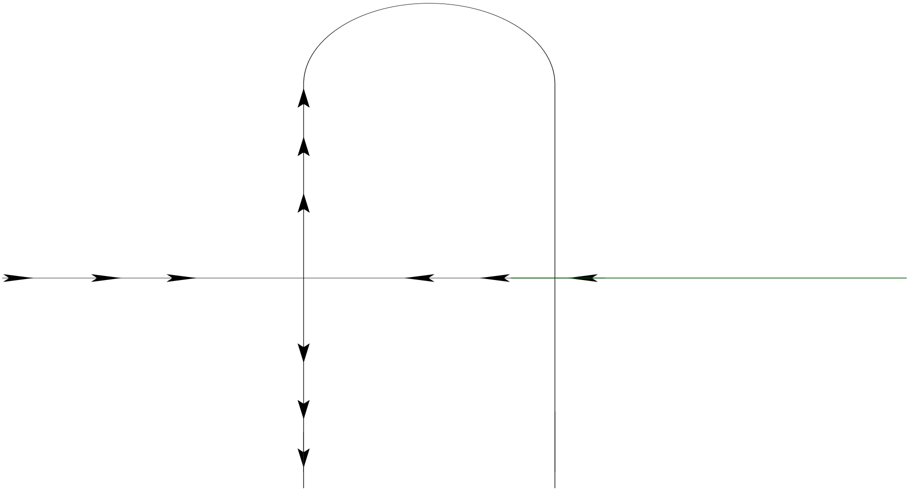
The Horseshoe

An example: $f : S^2 \longrightarrow S^2$ is a smooth map.

The Horseshoe



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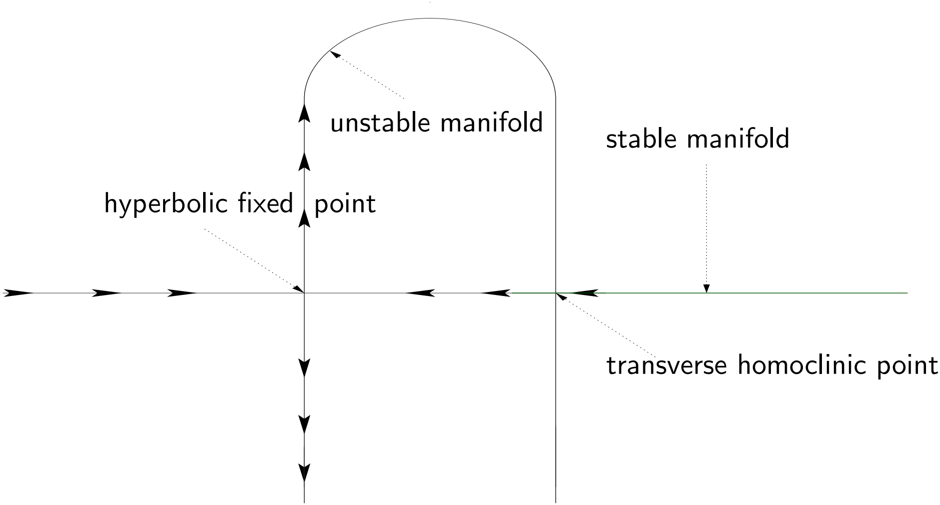

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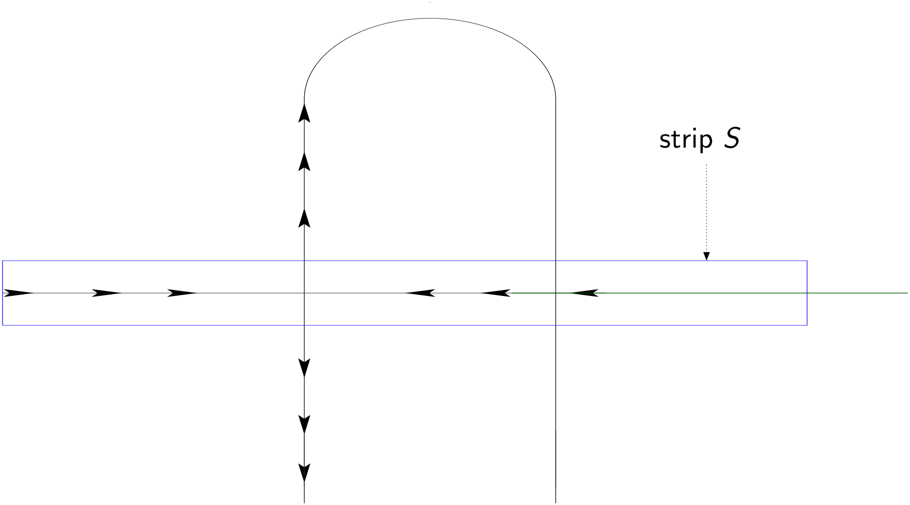

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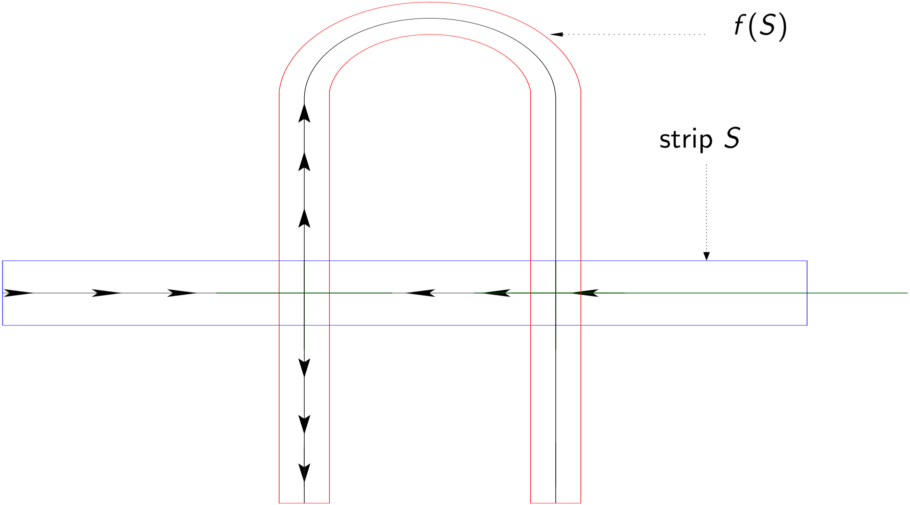

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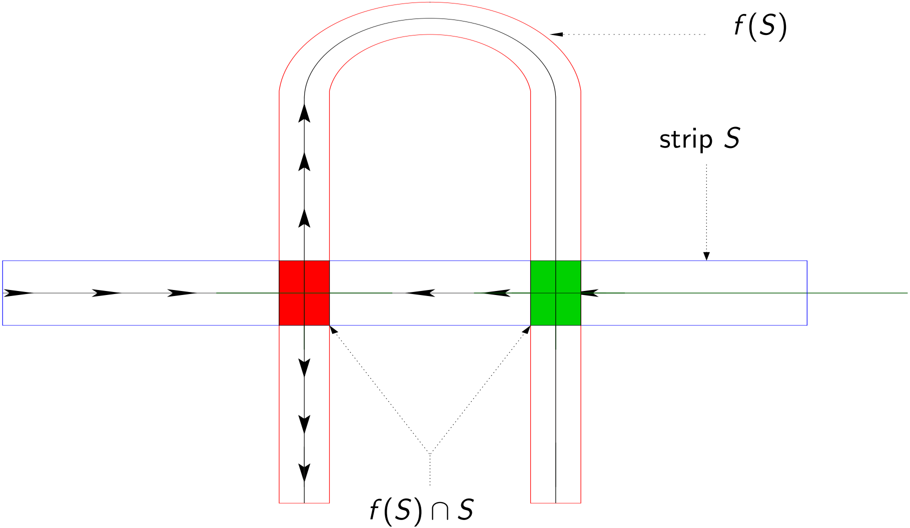

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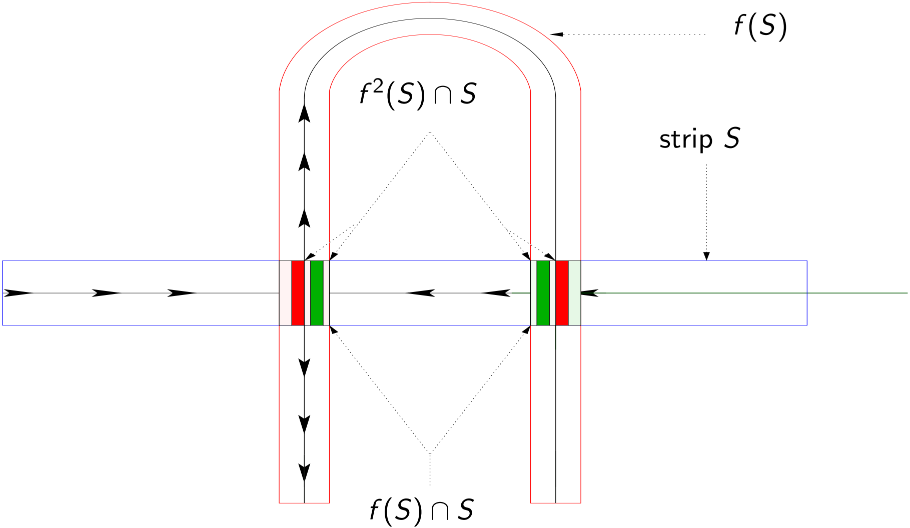

$f(S)$

$f^2(S) \cap S$

strip S

$f(S) \cap S$

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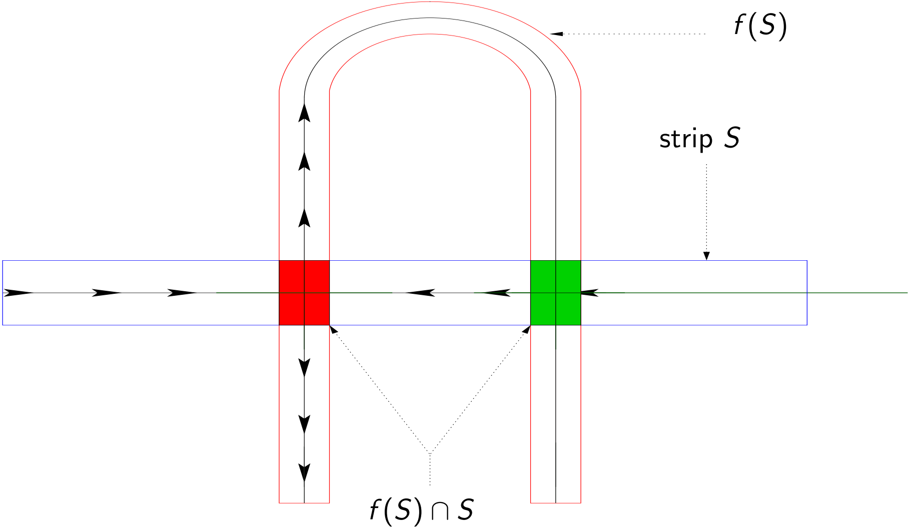

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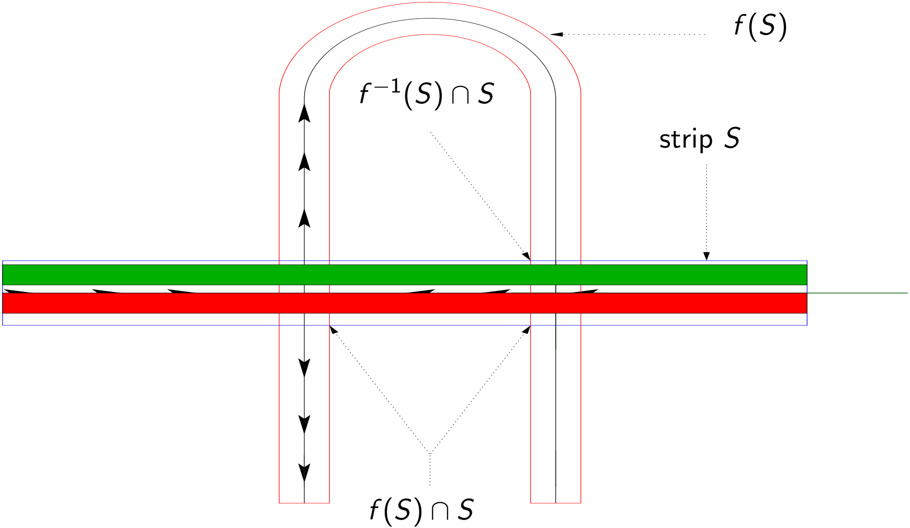

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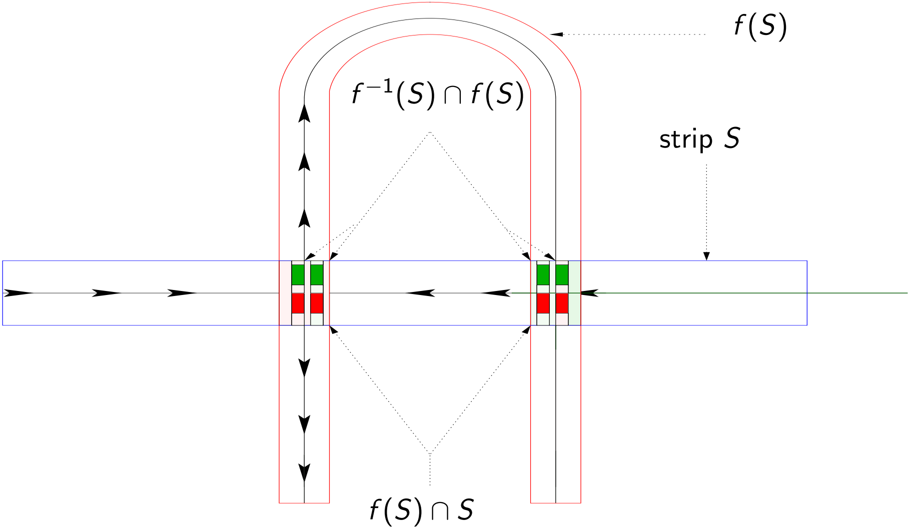

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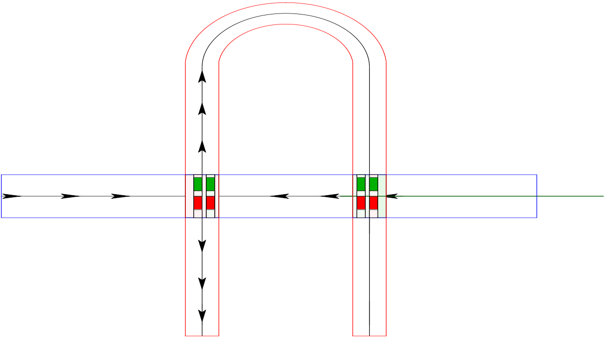

Implications:

1. Invariant Cantor set Λ
2. Invariant analytic function is constant on Λ
3. Invariant analytic function is constant.

Hamiltonian mechanics

Classical Newtonian Mechanics

$$F = ma.$$

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Equivalent to Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial x}$$

$$H = \text{Kinetic plus Potential Energy}$$

Hamiltonian mechanics

Poisson bracket $\{,\}$:

$$\{F, H\} = \frac{\partial F}{\partial x} \cdot \frac{\partial H}{\partial p} - \frac{\partial H}{\partial x} \cdot \frac{\partial F}{\partial p}$$

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on

$$T^*M = \{(x, p) \ : \ x \text{ is a point on } M, p \text{ is a momentum vector }\}.$$

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Theorem (Liouville-Arnold-Mineur)

Let $H = F_1, \cdots, F_n$ be n Poisson commuting functions. If

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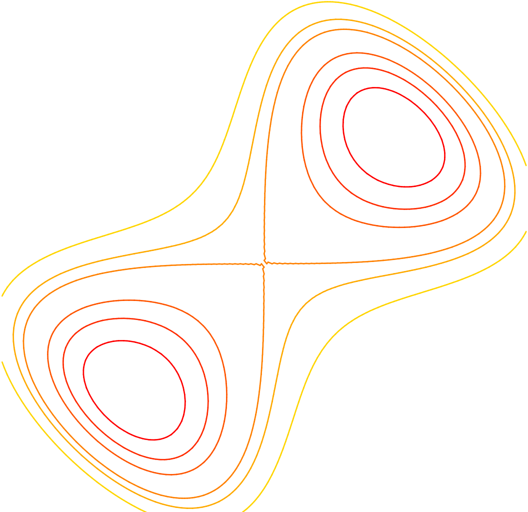
- *there are coordinates (θ, I) such that*
  - *each $F_j = F_j(I)$;*
  - *each vector field is linear*

$$\dot{\theta} = \frac{\partial F_j(I)}{\partial I} \qquad \dot{I} = 0.$$

Integrability

- A typical phase portrait:



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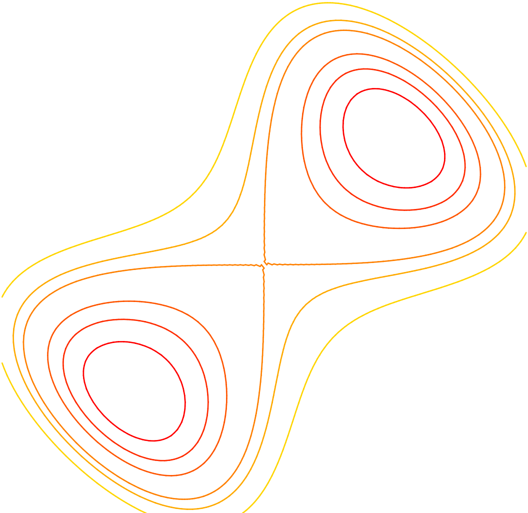

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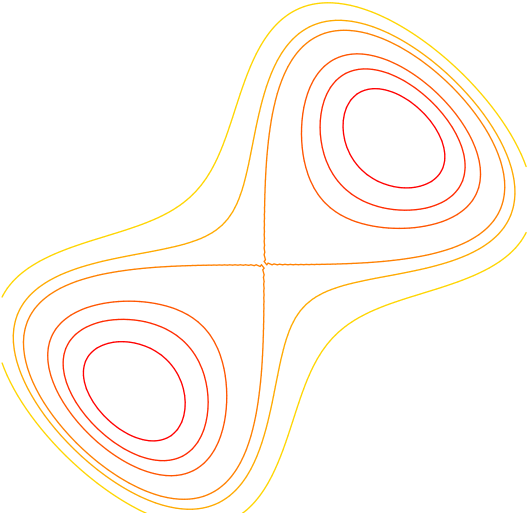

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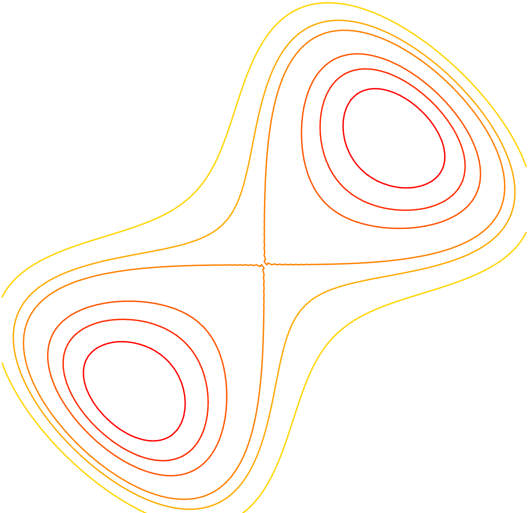

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- degenerations controlled by Morse-like behaviour

▸ To horseshoe ▸ To L-A-M

Questions

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Flat metrics: determine manifold almost completely;
Morse functions: information but no restrictions;
Positive Ricci curvature: restrictions but quite ‘flabby’.

Topology, Integrability and Entropy

Theorem (Fomenko with Zieschang, Matveev)

If $H : (M^4, \omega) \to \mathbf{R}$ *is integrable with a non-degenerate (or real-analytic) integral* F*, then*

1. *the regular levels of* H *are graph manifolds;*
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Graph manifolds

Building Blocks

1. $S_k = $ 2-disk with k open disks removed;
2. $M_k = S_k \times S^1$ or $S_k \tilde{\times} S^1$;
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Examples

1. $S^2 \times S^1 = D^2 \times S^1 \cup_{id} D^2 \times S^1$
2. $S^3 = D^2 \times S^1 \cup_{id} S^1 \times D^2$
3. $\mathbf{T}^3$,...
4. no hyperbolic manifolds

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If $H : (M^{2n}, \omega) \to \mathbf{R}$ *is completely integrable with non-degenerate integrals, then its topological entropy vanishes.*

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$$\begin{aligned} 2H &= p_x^2 + e^{2x} p_{y_0}^2 + e^{-2x} p_{y_1}^2 \\ I_1 &= p_{y_0}, \qquad I_2 = p_{y_1}. \end{aligned}$$

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- Discrete group: $u = \frac{1+\sqrt{5}}{2}$

$$\Delta = \{(u^k, \mathbf{y}) \, : \, y_0 = \frac{n + m\sqrt{5}}{2}, y_1 = \frac{n - m\sqrt{5}}{2}, \; k, m, n \in \mathbf{Z} \}.$$

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The hamiltonian $H : T^*(Sol/\Delta) \to \mathbf{R}$ *is completely integrable. Its flow has positive topological entropy.*

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Proof.

The integrals are

$$2H = p_x^2 + e^{2x} p_{y_0}^2 + e^{-2x} p_{y_1}^2,$$

$$I = p_{y_0} p_{y_1}, \qquad J = \exp(-I^{-2}) \times \sin \frac{2\pi \ln |p_{y_1}|}{\ln u}.$$

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Anosov diffeomorphism on $\{p_{y_1} = p_{y_2} = 0, p_x = 1\}$:

$$\begin{array}{rclcrcll} x & = & x_0 + t, & \qquad & p_x & = & 1, & \\ y_0 & = & y_{0,0}, & & p_{y_0} & = & 0, & \\ y_1 & = & y_{1,0}, & & p_{y_1} & = & 0, & \mod \Delta. \end{array}$$

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At $t = \ln u$

$$\begin{array}{rclrcl} x & \equiv & x_0, & p_x & \equiv & 1, \\ y_0 & \equiv & u^{-1} y_{0,0}, & p_{y_0} & \equiv & 0, \\ y_1 & \equiv & u y_{1,0}, & p_{y_1} & \equiv & 0, \quad \bmod \Delta. \end{array}$$

or

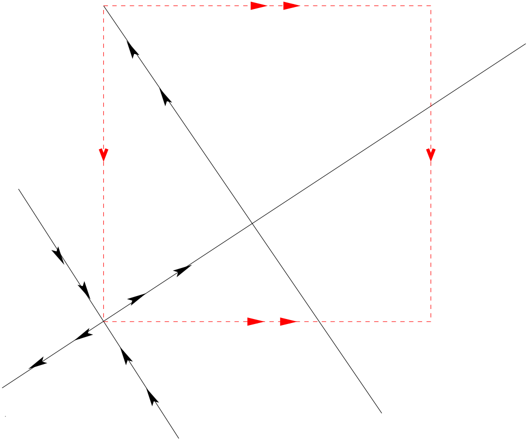
$$\mathbf{y}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{y}(0) \bmod \mathbf{Z}^2.$$

The Anosov Diffeomorphism

$$\mathbf{y} \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{y} \bmod \mathbf{Z}^2.$$

▸ To horseshoe

▸ To integrals



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8 model geometries in 3 dimensions:

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$S^2 \times \mathbf{E}^1$ $\mathbf{H}^2 \times \mathbf{E}^1$

S^3 $\widetilde{SL_2\mathbf{R}}$

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$$Nil$$

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Conjecture (Thurston)

Each compact 3-manifold admits a canonical decomposition into complete, finite-volume geometric 3-manifolds.

Results in 3 degrees of freedom

Definition (Semisimplicity)

Let

$$\begin{tikzcd}
\mathbf{T}^n \arrow[r, hook] & L \arrow[r, hook, "incl."] \arrow[d, "f"] & T^*M \arrow[r, two heads, "\pi"] & M \\
 & B & &
\end{tikzcd}$$

be a lagrangian fibration. If $\Gamma = T^*M - L$ *is a closed, nowhere dense tamely-embedded polyhedron, then we say* (f, L, B) *is* **semisimple**.

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Theorem (B. 2005)

Assume the GC. A compact 3-manifold admits a riemannian metric with a semisimple geodesic flow iff it admits one of five geometric structures:

$$\mathbf{E}^3, \qquad S^2 \times \mathbf{E}^1, \qquad S^3, \qquad Nil, \qquad Sol.$$

Sketch of proof

$\Rightarrow$ There is a subspace $L_0 \subset L$ s.t. $\pi_1(L_0) \to \pi_1(M)$ is almost onto and

$$\begin{tikzcd}
\mathbf{T}^3 \arrow[r, hook] & L_0 \arrow[r] \arrow[d, "f_0 = f|L_0"] & M \\
 & B_0 &
\end{tikzcd}$$

is a lagrangian fibration over a surface B_0.

Sketch of proof

$\Rightarrow$

1. There is a subspace $L_1 \subset L_0$ s.t. $\pi_1(L_1) \to \pi_1(M)$ is almost onto and

$$\begin{tikzcd}
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& B_1 \simeq S^1 &
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4. Otherwise, (Agol) GC implies that M admits $S^2 \times \mathbf{E}$ geometry.

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The real-analytic complete integrability of the geodesic flows of the compact geometries modeled on $\mathbf{E}^3$, $S^2 \times \mathbf{E}$, S^3 is well-known.

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The semisimplicity of the geodesic flows of *Nil*-geometry is proven by (B.) and that of *Sol* by (Bolsinov-Taimanov). ▸ To B-T ▸ Theorem

Exotic spheres and Tori

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$$\sum_{i=0}^{n} H^i(\mathbf{T}^n; \Gamma_i) \qquad \Gamma_i = \begin{cases} \text{group of exotic } i\text{-spheres}, & i \neq 4, \\ 0 & i = 4. \end{cases}$$

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Exotic tori and integrability

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Proof.

Claim: real-analytic integrability implies there is lagrangian torus F s.t. ρ is a diffeomorphism:

$$\begin{tikzcd}
F \arrow[r, hook, "\text{incl.}"] \arrow[dr, "\rho"'] & T^*\Sigma \arrow[d, "\pi"] \\
 & \Sigma.
\end{tikzcd}$$

We show $\deg \rho \neq 0$, use a result of Viterbo's that $\deg \rho \neq 0$ implies ρ is a covering map, then use a cohomology argument to show ρ is $1-1$. ☐

Witten-Kreck-Stolz spaces

Let $k, l \neq 0$ be coprime integers. Let

$$U = \left\{ \begin{bmatrix} z^k & 0 \\ 0 & z^l \end{bmatrix} \ : \ |z| = 1 \right\} \qquad \subset U_2 \times U_3.$$

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The riemannian metric on $M_{k,l}$ induced by the round metrics on $S^3 \times S^5$ is real-analytically completely integrable.

Questions

1. Are there completely integrable geodesic flows on exotic tori? Must these have positive entropy?
2. Is it possible to give a description of the smooth invariant which determines whether or not a topological manifold admits a semisimple geodesic flow?
3. Must a real-analytically integrable system in dimensions 6 and more have zero entropy?