A construction of integrable systems with positive entropy

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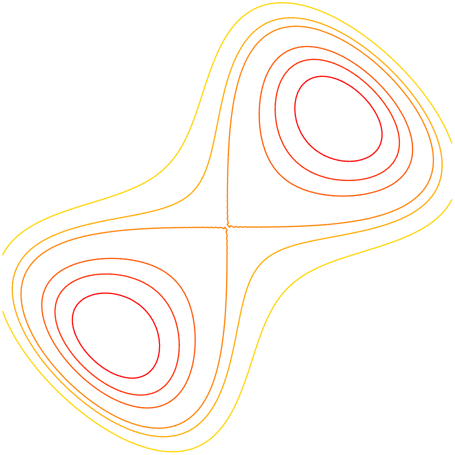
Integrability

- A typical phase portrait:



- fibred by tori
- tori degenerate
- degenerations controlled by Morse-like behaviour

Topology, Integrability and Entropy

Theorem (Fomenko with Zieschang, Matveev)

If $H : (M^4, \omega) \to \mathbf{R}$ *is integrable with a non-degenerate (or real-analytic) integral* F*, then*

1. *the regular levels of* H *are graph manifolds;*
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Theorem (Paternain, Moser)

In addition,

3. *the topological entropy vanishes.*

What about 3 or more degrees of freedom?

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Theorem (Paternain)

If $H : (M^{2n}, \omega) \to \mathbf{R}$ is completely integrable with non-degenerate integrals, then its topological entropy vanishes.

What about 3 or more degrees of freedom?

Theorem (Taimanov 1988)

Let (M^n, g) be a compact, real-analytic manifold. If the geodesic flow is real-analytically integrable, then

1. $b_1(M) \leq n$;
2. $\pi_1(M)$ *is almost abelian;*
3. *there is an injection* $H^*(\mathbf{T}^{b_1}) \hookrightarrow H^*(M)$.

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1. Surfaces: $S^2, \mathbf{T}^2$;
2. 3-manifolds:

$b_1 = 0$: $\tilde{M}$ is a homotopy S^3;
$b_1 = 1$: M is finitely covered by a homotopy $S^1 \times S^2$;
$b_1 = 2$: can't happen (Reidemeister);
$b_1 = 3$: M is finitely covered by a homotopy $\mathbf{T}^3$.

Suspension manifolds

$A \in \mathrm{GL}(2; \mathbf{Z})$, for example

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{or better } A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

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Σ_A is the suspension manifold of A:

$$\Sigma_A : \qquad \mathbf{R} \times \mathbf{T}^2 / \simeq \qquad \text{where } (0, A\theta) \simeq (1, \theta).$$

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Theorem (Bolsinov & Taimanov 2000)

There is a riemannian metric $\mathbf{g}$ *on* Σ_A *whose geodesic flow is completely integrable. Its flow contains the suspension flow of* A *as a subsystem.*

Sketch of the Proof

Proof.

1. identify universal cover of Σ_A with a Lie group
2. construct metric and integrals on universal cover
3. verify the construction descends
4. exhibit the suspension flow as a subsystem

$\square$

A Homogeneous Space

Universal Cover of Σ_A: $Sol = \mathbf{R} \star \mathbf{R}^2$

$$x.\mathbf{y} = (e^x y_0, e^{-x} y_1).$$

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$$\Delta = \{(u^k, \mathbf{y}) \ : \ y_0 = \frac{n + m\sqrt{5}}{2}, y_1 = \frac{n - m\sqrt{5}}{2}, \ k, m, n \in \mathbf{Z} \ \}.$$

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Lemma

$$\Sigma_A = \Delta \backslash Sol.$$

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On $T^*\Sigma_A$:

$$I = p_{y_0} p_{y_1} \qquad\qquad J = \exp(-I^{-2}) \times \sin\left[\frac{2\pi \ln|p_{y_0}|}{\ln u}\right]$$

Taimanov theorem

The Suspension Flow

The shortest path from $(0, \theta)$ to $(1, \theta)$ with

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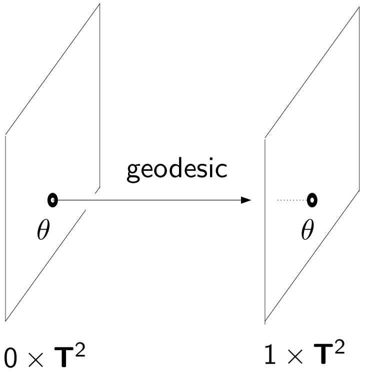

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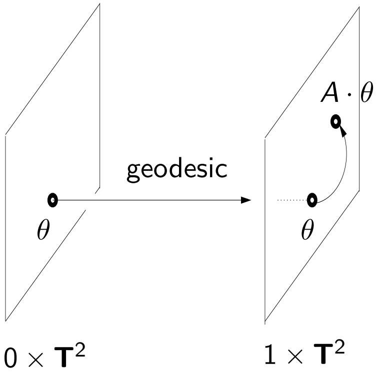

The momentum map

$$Sol = \mathbf{R} \star \mathbf{R}^2 : \qquad (0, y_0', y_1') \cdot (x, y_0, y_1) = (x, y_0' + y_0, y_1' + y_1)$$

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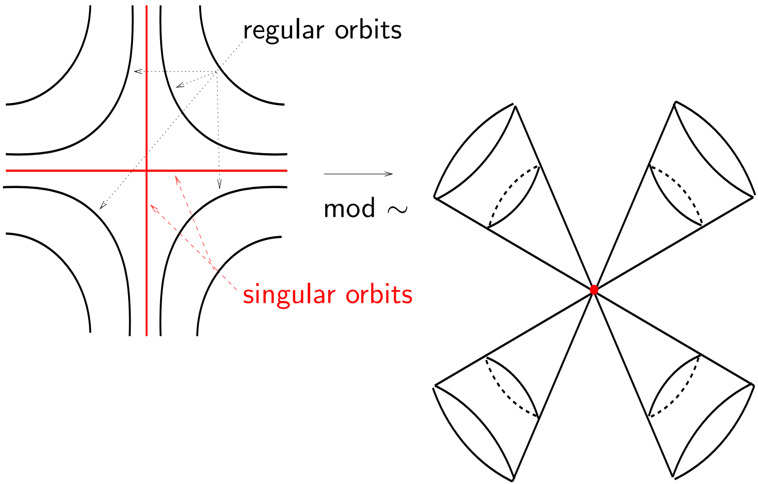
$$\begin{tikzcd}
(x, y_0, y_1, p_x, p_{y_0}, p_{y_1}) \arrow[r, mapsto] \arrow[d, dotted] & (p_{y_0}, p_{y_1}) \arrow[d, dotted] \\
T^*Sol \arrow[r, "\lambda"] \arrow[d, "\text{mod } \Delta"'] & \mathrm{Lie}(\mathbf{R}^2)^* \arrow[d, "\text{mod } \frac{\Delta}{\Delta \cap \mathbf{R}^2}"] \\
T^*\Sigma_A \arrow[r, "\lambda'"] & \mathrm{Lie}(\mathbf{R}^2)^* \Big/ \frac{\Delta}{\Delta \cap \mathbf{R}^2} \\
\Delta \cdot (x, y_0, y_1, p_x, p_{y_0}, p_{y_1}) \arrow[r, mapsto] \arrow[u, dotted] & (p_{y_0}, p_{y_1}) / \sim \arrow[u, dotted]
\end{tikzcd}$$

where $(e^{\mu} p_{y_0}, e^{-\mu} p_{y_1}) \sim (p_{y_0}, p_{y_1})$.

Orbits in $\mathrm{Lie}(\mathbf{R}^2)^*/\sim$



reduced space: $\mathrm{Lie}(\mathbf{R}^2)^*/\sim$

Ingredients of a generalisation

Configuration space

suspension manifold

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Riemannian metric Toda lattices

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Configuration space	suspension manifold
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Configuration space	suspension manifold
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Invariance	equivariance

Suspension manifolds of abelian subgroups of $\mathrm{GL}(b; \mathbf{Z})$

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$$u \cdot (x, \theta) = (-u + x, u \cdot \theta) \qquad \forall u \in A, x \in A_{\mathbf{R}}, \theta \in \mathbf{T}^b.$$

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Definition

The manifold

$$\Sigma_A = \mathbf{R}^a \times \mathbf{T}^b / A$$

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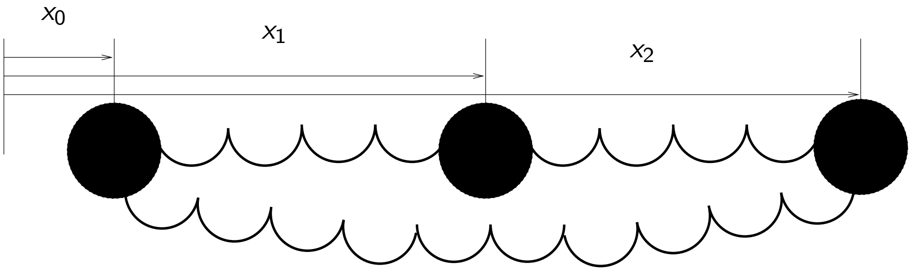
The $\mathbf{R}^a \times 0$ action on $\mathbf{R}^a \times \mathbf{T}^b$ induces a natural action on $\Sigma_A \implies$ suspension flow of A.

Toda Lattices



Potential energy:

$$V = \exp(x_0 - x_1) + \exp(x_1 - x_2) + \exp(x_2 - x_0)$$

Toda Lattices

A non-linear coupled oscillator

$$\ddot{x}_i = \exp(x_i - x_{i+1}) - \exp(x_{i-1} - x_i), \qquad i = 1, \ldots, n \bmod n. \qquad (*)$$

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Flaschka Transform

$$a_i := \epsilon_i \exp(x_i/2 - x_{i+1}/2), \qquad b_i := \dot{x}_i,$$

$$L := \begin{bmatrix} b_1 & a_1 & 0 & \cdots & 0 & \lambda a_n \\ a_1 & b_2 & a_2 & \cdots & 0 & 0 \\ 0 & a_2 & b_3 & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & \ddots & b_{n-1} & a_{n-1} \\ \lambda^{-1} a_n & 0 & & & a_{n-1} & b_n \end{bmatrix}$$

$$M := 1/2 \times (L_+ - L'_+)$$

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$$M := 1/2 \times (L_+ - L_+')$$

Toda Lattices and Positive-Entropy Integrable Systems

Theorem (B. 2004, B. 2008)

Let $A < \mathrm{GL}(b; \mathbf{Z})$ *be a maximal torsion-free abelian group of semisimple elements. There are convex, polynomial-in-momenta hamiltonians on* $T^*\Sigma_A$ *such that*

1. *for each periodic Toda lattice* Ψ *of rank* n*, there is an integrable hamiltonian* $H_\Psi : T^*\Sigma_A \to \mathbf{R}$*;*
2. *the flow of* H_Ψ *contains the suspension flow of all* $u \in A$*;*
3. *for* $a = 1$*,* $b = 2$ *and* $A < \mathrm{GL}(2; \mathbf{Z})$ *is cyclic, the* $A_1^{(1)}$ *Toda lattice yields the Bolsinov-Taimanov example.*

Idea of a Proof: Use the Flaschka transform

$\Sigma_A = \Delta \backslash \mathbf{S}$ where $\mathbf{S} = \mathbf{R}^b \star A_{\mathbf{R}}$ is a solvable Lie group and $\Delta = \mathbf{Z}^b \star A$ acts by left-translation:

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The momentum map λ of $\mathbf{R}^b$'s left action and
the momentum map ρ of $\mathbf{S}$'s right action

$$\begin{tikzcd} \mathrm{Lie}(\mathbf{R}^b)^* & T^*\mathbf{S} \arrow[l, two heads, "\lambda"'] \arrow[r, two heads, "\rho"] \arrow[rr, bend right, two heads, "\text{Lax matrix}"'] & \mathrm{Lie}(\mathbf{S})^* \arrow[r, two heads, "\text{F.T.}"] & T^*\mathbf{R}^a \end{tikzcd}$$

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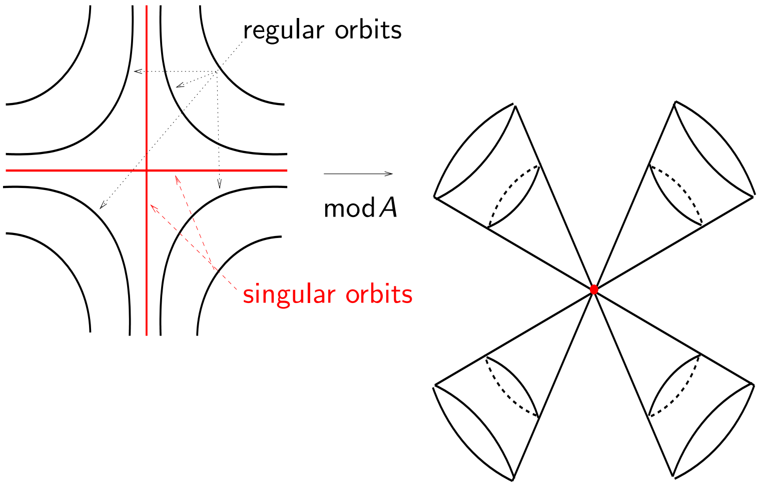
We use equivariance to reduce λ. Works since $\mathrm{Lie}(\mathbf{R}^b)^*/A$ is a smooth manifold except for codim-1 singular bit.

Orbits in $\mathrm{Lie}(\mathbf{R}^b)^*/A$



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the Toda flow factors onto the geodesic flow of a flat riemannian metric.

The lift of the geodesic $t \mapsto (x + t \cdot u, u)$ on $T^*\mathbf{T}^a$ embeds the suspension flow of $u \in A$.

Topological Conjugacy

The Flaschka transform is non-natural, it depends on a permutation σ of $a+1$ roots:

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Question

Are each of the Toda-type hamiltonian flows topologically conjugate?

Topological Conjugacy

Definition

$A < \mathrm{GL}(b;\mathbf{Z})$ is *totally hyperbolic* if its number-theoretic normal closure has no non-trivial elements with eigenvalues on S^1.

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Theorem

Let $\mathrm{h} : A \to \mathbf{R}$ *be the entropy functional:*

$$\mathrm{h}(u) = h_{top}(u) \qquad\qquad u \in A$$

If the flows φ_1 *and* φ_2 *are topologically conjugate on their unit sphere bundles, then there is* $f \in \mathrm{Aut}(A)$ *such that*

$$\mathrm{h} \circ f = \mathrm{h}.$$

If A *is totally hyperbolic, then* f *is induced by a Galois automorphism.*

Numbers of Topological Conjugacy Classes

$b-1$=rank=2	$A_2^{(1)}$	$C_2^{(1)}/A_{2\cdot2}^{(2)}/D_{2+1}^{(2)}$	$G_2^{(1)}/D_4^{(3)}$		
$\mathrm{Aut}(F/\mathbf{Q})=1$	1	3	6		
$\mathrm{Aut}(F/\mathbf{Q})=\mathbf{Z}_3$	1	1	2		
$b-1$=rank=3	$A_3^{(1)}$	$C_3^{(1)}/A_{2\cdot3}^{(2)}/D_{3+1}^{(2)}$	$B_3^{(1)}/A_{2\cdot3-1}^{(2)}$		
$\mathrm{Aut}(F/\mathbf{Q})=1$	3	12	12		
$\mathrm{Aut}(F/\mathbf{Q})=\mathbf{Z}_2\oplus\mathbf{Z}_2$	3	6	3		
$\mathrm{Aut}(F/\mathbf{Q})=\mathbf{Z}_4$	2	4	3		
$b-1$=rank=4	$A_4^{(1)}$	$C_4^{(1)}/A_{2\cdot4}^{(2)}/D_{4+1}^{(2)}$	$B_4^{(1)}/A_{2\cdot4-1}^{(2)}$	$D_4^{(1)}$	$F_4^{(1)}/E_6^{(2)}$
$\mathrm{Aut}(F/\mathbf{Q})=1$	12	60	60	5	120
$\mathrm{Aut}(F/\mathbf{Q})=\mathbf{Z}_5$	4	12	12	1	24
$b-1$=rank=5	$A_5^{(1)}$	$C_5^{(1)}/A_{2\cdot5}^{(2)}/D_{5+1}^{(2)}$	$B_5^{(1)}/A_{2\cdot5-1}^{(2)}$	$D_5^{(1)}$	
$\mathrm{Aut}(F/\mathbf{Q})=1$	60	360	360	90	
$\mathrm{Aut}(F/\mathbf{Q})=\mathbf{Z}_6$	14	64	60	17	
$\mathrm{Aut}(F/\mathbf{Q})=S_3$	19	72	60	21	
$b-1$=rank=6	$A_6^{(1)}$	$C_6^{(1)}/A_{2\cdot6}^{(2)}/D_{6+1}^{(2)}$	$B_6^{(1)}/A_{2\cdot6-1}^{(2)}$	$D_6^{(1)}$	$E_6^{(1)}$
$\mathrm{Aut}(F/\mathbf{Q})=1$	360	2520	2520	630	840
$\mathrm{Aut}(F/\mathbf{Q})=\mathbf{Z}_7$	54	360	360	90	120
$b-1$=rank=7	$A_7^{(1)}$	$C_7^{(1)}/A_{2\cdot7}^{(2)}/D_{7+1}^{(2)}$	$B_7^{(1)}/A_{2\cdot7-1}^{(2)}$	$D_7^{(1)}$	$E_7^{(1)}$
$\mathrm{Aut}(F/\mathbf{Q})=1$	2520	20160	20160	5040	20160
$\mathrm{Aut}(F/\mathbf{Q})=\mathbf{Z}_8$	332	2544	2520		

$F=\mathbf{Q}(\lambda)$ for some eigenvalue λ of a primitive element in A.

Thank you

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Thank you for your attention.